INFERENCE FOR TWO-PARAMETER EXPONENTIALS UNDER TYPE I CENSORING

By

LILY LLORENS MANTELLE

A DISSERTATION PRESENTED TO THE GRADUATE SCHOOL
OF THE UNIVERSITY OF FLORIDA IN
PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1986

ACKNOWLEDGEMENTS

I wish to thank my advisor at the University of Florida, Dr. Malay Ghosh, for all his help and encouragement as a professor and as a friend in putting this manuscript together. I also wish to thank Dr. P.V. Rao, Dr. Kenneth M. Portier, Dr. Andrew Rosalsky and Dr. Boghos D. Sivazlian for serving on my committee.

Secondly, I want to thank my husband, my daughter and my mother for their patience, support and understanding.

Lastly I want to express my gratitude to Monroe A. Crews for doing a good job in typing this manuscript.

TABLE OF CONTENTS

Abstract of Dissertation Presented to the Graduate School of the University of Florida in Partial Fulfillment of the Requirements for the Degree of Doctor of Philosophy

INFERENCE FOR TWO-PARAMETER EXPONENTIALS UNDER TYPE I CENSORING

By

LILY LLORENS MANTELLE

December 1986

Chairman: Dr. Malay Ghosh
Major Department: Statistics

This investigation considers estimation and hypothesis testing involving parameters of one or more location and scale parameter exponentials under Type I censoring when sampling is done both with and without replacement.

For sampling with replacement, the class of unbiasedly estimable parametric functions is completely characterized. It turns out, as a consequence of our general results that quite often neither the location nor the scale parameter is unbiasedly estimable. The failure rate admits an unbiased estimator in some special cases. Maximum likelihood estimators (MLEs) of the location parameters and failure rates are also obtained. Then for the location parameters a modified MLE is proposed which achieves asymptotically a 50% mean square error reduction and a 100% bias reduction over the usual MLEs. Asymptotic normality as well as

mean square convergence results are established for the MLEs of the scale parameters.

A parallel investigation is conducted for the without replacement case and similar results are obtained. However, in this case, only a partial characterization of estimable parametric functions is obtained, and it is shown also that neither the location nor the scale parameter admits an unbiased estimator.

Generalized Likelihood Ratio Tests (GLRT) for the equality of the location and/or failure rates of several independent location and scale parameters exponentials are also considered, when sampling is done both with and without replacement. For testing the equality of the failure rates, asymptotic distributions of the GLRT criteria are obtained both under the null hypothesis and under local alternatives. For testing the equality of the location parameters, asymptotic null distributions of the GLRT criteria are obtained.

CHAPTER ONE

INTRODUCTION

1.1 Background and Previous Research

This investigation considers estimation and hypothesis testing involving parameters of one or more location and scale parameters exponentials under Type I censoring. Under such censoring, an experiment consisting of putting units to test independently until they fail is stopped after a fixed amount of time. This is in contrast with Type II censoring where experimentation stops after a fixed number, say r of failures.

The above censoring schemes are most suitable for drawing inferences based on lifetime data. Such data are most commonly encountered in engineering and medical sciences. With the usual life testing terminology, the events of interest are usually referred to as failures, and the mean rate of failure is referred to as the failure rate.

The location-scale exponential distribution is very often used to model the lifetimes of manufactured items. Such a distribution has pdf

$$f(x) = \zeta \exp[-\zeta(x-\eta)] I_{[x>\eta]}, \qquad (1.1.1)$$

where $\eta \in (0,\infty)$ is the location parameter, and $\zeta \in (0,\infty)$ is the scale parameter and $I_A = 1$ when A happens and is zero otherwise. The reciprocal ζ^{-1} is called the scale parameter and shall be

denoted by θ. It is either suspected or known that there is a "failure free" period before the first failure is observed, and this justifies the inclusion of the location parameter η. The parameter space $(0,\infty)$ for η is justified on the ground that the starting time of an experiment is conventionally taken as zero. However, no technical difficulty is encountered if one considers instead the parameter space $(-\infty,\infty)$ for η. In the life-testing terminology, η is also referred to as the guarantee time or the threshhold parameter. If t (>0) denotes the censoring time or fixed duration of the experiment, it is assumed that $\eta < t$ since otherwise no failures will occur.

Most of the inference problems concerning η and ζ^{-1} are usually directed towards either complete (uncensored) data or Type II censored data. In such situations, detailed discussion for estimation, hypothesis testing or confidence intervals in the one sample case appears in Mann, Schafer and Singpurwalla (1974). However, for Type I censored data, literature is not at all that extensive. The hypothesis testing problem for η is addressed in Wright, Engelhardt and Bain (1978) for the one sample case (see also Bain (1978)).

Before proceeding further, it is important to distinguish between two modes of sampling, namely sampling with and without replacement. In the former case, an item failing before termination of the experiment, is either repaired or replaced by a similar new item. This does not happen in the other case.

For an example of Type I censoring with replacement, consider the result of a fatigue test as conducted by Butler and Rees (1974) to determine the suitability of various metals for aircraft construction. In one phase of study, titanium and steel specimens were tested for crack initiation due to fatigue. Each specimen was subjected to stresses in varying amounts and patterns similar to those occurring in flight. These stress patterns were repeated until a crack was detected and the total number of load cycles (which can be thought of as a laboratory measure of flight time) until crack detection was recorded. Then the crack was repaired, and the test was resumed until another crack was detected, the total number of load cycles to this failure was recorded etc., Also, the chi-squared goodness-of-fit test indicated for their data that the exponential distribution provided a reasonable model for the interfailure times (i.e. the times between failures) in the range that the tests were conducted.

For testing without replacement, the following example was given in Wilk, Gnanadesikan and Huyett (1962) and Wright et al. (1978). Consider the failure times (in weeks) from an accelerated life test of several transistors. The censoring time (measured in weeks) was 40. Since the failed transistors were not replaced, this was an example of sampling without replacement. Engelhardt and Bain (1975) showed that the exponential model was reasonable for these data.

For Type I censoring, first consider the with replacement case. Suppose n items are put to test, and the lifetimes of these items are iid with common pdf given in (1.1.1). It is assumed that the lifetimes of repaired or a replacement item is exponential with the same failure rate ζ, but with location para-meter equal to zero. This assumption is appropriate in many instances because a repaired item would not be expected to have a failure free period again. Even if defective parts were replaced, such an assumption might be reasonable if the original parts or system were sealed or treated in a special manner. Also, lifetimes of original and replacement parts are assumed to be independent.

To derive the joint distribution of the ordered failure times, one proceeds as follows. Let Y_{ij} denote the elapsed time for the j^{th} unit between (i-1)st and ith failure. Note that by the memoryless property of the exponential pdf we don't need to consider how long a unit has lived when constructing these intervals. Hence $Y_{11},\ldots,Y_{1n}$ are iid with pdf given in (1.1.1), while the remaining Y_{ij}'s are iid with location parameter $\eta = 0$ and failure rate ζ. Now if $Y_i = \min\limits_{1 \leq j \leq n} Y_{ij}$ $(i=1,2,\ldots)$, then Y_i's are the interfailure times of the experiment and they are independent with pdf of Y_1 given as

$$f(y_1) = (n\zeta)\exp\left(-n\zeta(y_1-\eta)\right) I_{[y_1>\eta]}, \tag{1.1.2}$$

while $Y_2,Y_3,\ldots$ are iid with common pdf

$$f(y) = (n\zeta)\exp(-n\zeta y) I_{[y>0]}. \tag{1.1.3}$$

The ideas previously described are further clarified in Figure 1.1 for $n = 5$, $i = 2$ and assuming four failures occurred. Also, in Figure 1.1 we use X to denote a failed unit which is instaneously repaired or replaced by a similar item and we let $x_{(1)},\dots,x_{(4)}$ denote the four ordered failure times.

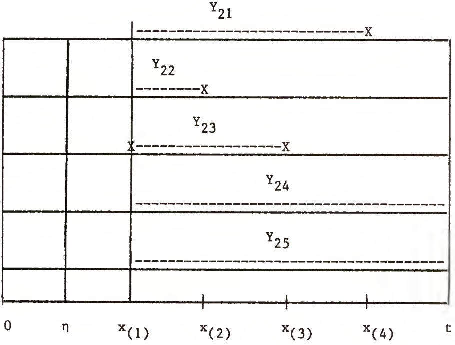

FIGURE 1.1: Clarification of Interfailure Times.

(Note above that $Y_2 = \min_{1<j<4} X_{2j}$).

The number of failures in the interval (η,t) is a random variable which we denote by R. The first lemma of this section gives the distribution of R.

Lemma 1.1.1 Suppose $Y_1, Y_2, \ldots$ are independent with pdf's given in (1.1.2) and (1.1.3). Then $R \sim$ Poisson $(n\zeta(t-\eta))$.

Proof The result follows from the definition of a Poisson process (see for example Barlow and Proschan (1981)).

The next lemma provides the conditional pdf of the ordered failure times given $R = r(>0)$

Lemma 1.1.2 Given $R = r > 0$, let $X_{(1)} < \ldots < X_{(r)}$ denote the ordered failure times. Then the joint conditional pdf of $X_{(1)},\ldots,X_{(r)}$ given $R = r > 0$ is

$$f(x_{(1)},\ldots x_{(r)}|r) = r!(t-\eta)^{-r}, \quad \eta < x_{(1)} < \ldots < x_{(r)} < t \qquad (1.1.4)$$

which is the joint distribution of the order statistic in a random sample of size r from the uniform (η,t) distribution.

Proof Note that given $R = r > 0$

$$\left(X_{(1)}, X_{(2)} - X_{(1)}, \ldots, X_{(r)} - X_{(r-1)}\right) \stackrel{d}{=} (Y_1, Y_2, \ldots, Y_r).$$

Hence, using the fact that $P(Y_i > y) = \exp(-n\zeta y)$ for $i \geq 2$

$$f(x_{(1)},\ldots,x_{(r)}|r)$$

$$= \left[(n\zeta)\exp\left(-n\zeta(x_{(1)} - \eta)\right) \prod_{i=2}^{r} \left\{(n\zeta)\exp\left(-n\,\zeta(x_{(i)} - x_{(i-1)})\right)\right\}\right.$$

$$\cdot\, P\left(Y_{r+1} > t - x_{(r)}\right) \div P(R = r)$$

$$= \left[(n\zeta)^r \exp\left(-n\zeta(t-\eta)\right)\right] / \left[\exp\left(-n\zeta(t-\eta)\right)\{n\zeta(t-\eta)\}^r\right] / r!$$

$$= r!\ (t-\eta)^{-r}. \quad \square$$

In view of Lemmas 1.1.1 and 1.1.2, the joint pdf of $X_{(1)},\ldots,X_{(R)}$ and R is given by

$$f(x_{(1)},\ldots,x_{(r)},r) = (n\zeta)^r \exp\left(-n\zeta(t-\eta)\right), \qquad \eta<x_{(1)}<\ldots<x_{(r)}<t,$$

$$r = 1,2,\ldots \qquad (1.1.5)$$

$$P(R = 0) = \exp\left(-n\zeta(t - \eta)\right). \qquad (1.1.5a)$$

The maximum likelihood estimators (MLEs) of η and ζ^{-1} based on the joint pdf given in (1.1.5) and (1.1.5a) are given in Bain (1978). But he did not study any properties of these estimators. Tests for η for unknown ζ^{-1} are given in Wright et al. (1978). Other than the above, we are not aware of any estimation or hypothesis testing study in Type I censoring with replacement from the location-scale exponential distribution.

For sampling without replacement, n items are put to test whose failure times are iid with pdf given in (1.1.1). In this situation, everytime an item fails, it is not replaced. We denote by $X_1, \ldots X_n$ the failure times. The ordered failure times are denoted as before by $X_{(1)} < \ldots < X_{(n)}$. We observe X_i only if it is less than or equal to t so that the number of failures in this case is given by $R = \Sigma_{i=1}^{n} I_{[X_i < t]}$, where $I_A = 1$ if the event A happens, and $I_A = 0$, otherwise. Thus, in this case $R \sim \text{Bin}\left(n,\ 1 - \exp(-\ \zeta(t - \eta))\right)$.

The joint pdf of the ordered failure times and R is obtained as follows.

Conditional on $R = r\ (>0)$ the joint distribution of $W_1 = X_{(1)} - \eta, \ldots, W_r = X_{(r)} - \eta$ is the same as that of r order statistics from a random sample of size r from an exponential distribution truncated at $t - \eta$. Now appealing to Theorem 2.2 in page 51 of Bain (1978), one gets, the joint pdf of $W_1, \ldots, W_r$

given R = r (>0) as

$$f(w_1,\ldots,w_r|r) = r!\prod_{i=1}^{r}\{\exp(-\zeta w_i)/(1-\exp(-\zeta(t-\eta)))\}$$

Hence,

$$f(x_{(1)},\ldots,x_{(r)}|r) = r!\prod_{i=1}^{r}\{\exp(-\zeta(x_{(i)}-\eta))/(1-\exp(-\zeta(t-\eta)))\} \qquad (1.1.6)$$

Since $R \sim \text{Bin}\,(n,(1-\exp(-\zeta(t-\eta))))$, it follows from (1.1.6) that the joint pdf of $X_{(1)},\ldots,X_{(r)}$ and R is

$$f(x_{(1)},\ldots,x_{(r)},r) = \frac{n!\zeta^r}{(n-r)!}\prod_{i=1}^{r}\{\exp(-\zeta(x_{(i)}-\eta))\}$$

$$\cdot\{\exp(-(n-r)\zeta(t-\eta)\} \qquad \text{for } r = 1,2,\ldots n; \qquad (1.1.7)$$

$$P(R=0) = \exp[-n\zeta(t-\eta)]. \qquad (1.1.7a)$$

In this situation, MLEs of η and ζ^{-1} are obtained in Bain (1978), and tests for η treating ζ^{-1} as a nuisance parameter are given in Wright et al.

Multisample extensions of the ideas described earlier are as follows. Suppose that for some $k \geq 2$ the experiment consists of putting $n_1,\ldots n_k$ items to test independently. Also, the lifetimes of all these items are independently distributed and the lifetimes of the items in the ith group have common pdf

$$f(x) = \zeta_i\exp(-\zeta_i(x-\eta_i))I_{[x\geq\eta_i]}, \quad i=1,\ldots,k \qquad (1.1.8)$$

Let t_i denote the censoring time for the ith group, and let R_i denote the number of failures occurring before time t_i $(i=1,\ldots,k)$. It is assumed that $\eta_i < t_i$ for all $i = 1,\ldots,k$. For sampling with replacement, within each group, an item failing before the censoring time is either repaired or replaced. Let

$X_{(i1)} < \cdots < X_{(ir_i)}$ denote the ordered failure times for the ith group when $r_i > 0$ ($i = 1,\ldots,k$). Note that for $r_i=0$, $\min\limits_{1<j<n_i} X_{ij} > t_i$. Define $S = \{i : r_i > 0\}$ and $\bar{S} = \{j : r_j = 0\}$

(S or $\bar{S}$ can be empty with positive probability). Then generalizing (1.1.5) and (1.1.5a), the joint pdf of $X_{(i1)},\ldots,X_{(iR_i)},R_i$ ($i = 1,\ldots,k$) is given by

$$f(x_{(11)},\ldots,x_{(1r_1)},r_1,\ldots,x_{(k1)},\ldots,x_{(kr_k)},r_k)$$

$$= \prod_{i\in S} \{(n_i\zeta_i)^{r_i} \exp(-n_i\zeta_i(t_i - \eta_i))I_{[\eta_i<x_{(i1)}<\cdots<x_{(ir_i)}<t_i]}\}$$

$$\cdot \{\prod_{i\in\bar{S}} \exp(-n_j\zeta_j(t_j-\eta_j))\} \tag{1.1.9}$$

For sampling without replacement, generalizing (1.1.7) and (1.1.7a), one gets the joint pdf of $X_{(i1)},\ldots,X_{(iR_i)},R_i$ ($i=1,2\ldots k$) given by

$$f(x_{(11)},\ldots,x_{(1r_1)},r_1,\ldots,x_{(k1)},\ldots,x_{(kr_k)},\ldots,r_k)$$

$$= \prod_{i\in S} \Big[\frac{n_i!\ \zeta_i^{r_i}}{(n_i - r_i)!} \exp(-\{\sum_{j=1}^{r_i}\zeta_i(x_{(ij)} - \eta_i)$$

$$+ (n_i-r_i)\zeta_i(t_i-\eta_i)\})I_{[\eta_i<x_{(i1)}<\cdots<x_{(ir_i)}<t_i]}\Big]$$

$$\cdot \prod_{j\in\bar{S}} [\exp(-n_j\zeta_j(t_j - \eta_j))]. \tag{1.1.10}$$

1.2 The Present Research

In this dissertation we address several estimation and hypothesis testing problems involving location and scale parameters of the exponential distribution in one and many sample situations. First consider Type I censoring with replacement. In this case, in Chapter Two we have characterized the class of unbiased estimators for functions involving the location and scale parameters for the one and two sample problem. The end conclusion is that for most of the parameters of interest, there does not exist any unbiased estimators. The results can be easily generalized to the multiple sampling case, but we have refrained from doing so to preserve algebraic simplicity. In Chapter Three we have considered maximum likelihood estimation of location and scale parameters in one and two sample cases. The asymptotic distribution as well as the mean squared error (MSE) of the MLE is obtained in this section. Also, an asymptotically unbiased estimator for the location parameter is proposed which achieves asymptotically 50% MSE reduction than the MLE. Next, in Chapter Four, we focus on generalized likelihood ratio tests (GLRT) for the equality of the location and/or the failure rates of k independent location and scale parameter exponential when censoring times for the k groups are possibly distinct. For testing the equality of the failure rates, asymptotic distributions of the GLRT criterion are obtained both under the null hypothesis and under local alternatives. For

testing the equality of the location parameters, asymptotic null distributions of the GLRT criteria are obtained.

From Chapter Five onwards, the without replacement case is considered. In this case, MLEs of the location and scale parameters as well as their asymptotic distributions are obtained both in the one and two sample cases. Modified MLEs for the location parameter achieving MSE reduction over the MLEs are also introduced. Unlike the with replacement case a complete characterization of parametric functions admitting unbiased estimators seems difficult here owing to the complexity of the distributions of the minimal sufficient statistics. It is shown however that neither the location nor the scale parameter is unbiasedly estimable either in the one or in the two sample cases.

Finally, in Chapter Six GLRTs are derived for testing the equality of the location parameters and/or the failure rates. The asymptotic distributions obtained are very similar to their counterparts in the with replacement situation.

CHAPTER TWO

UNBIASED ESTIMATION FOR THE WITH REPLACEMENT CASE

2.1 Introduction

In this chapter we consider unbiased estimation of functions of location and scale parameters of exponential distributions in one as well as two sample problems, where observations are censored in time and sampling is done with replacement.

Our objective is to characterize estimable functions (i.e. those which admit unbiased estimators) of location and scale parameters. The one-sample case is considered in Section 2.2. As a consequence of the main characterization result in this case, it follows that if both the location and scale parameters are unknown, any function involving only the location parameter is not estimable. In addition, neither the scale parameter nor its reciprocal (the failure rate) is estimable. However if the scale parameter is known, any differentiable function of the location parameter is estimable, while if the location parameter is known, any power series in the failure rate is estimable.

A similar investigation is pursued in Section 2.3 for the two sample case. Several cases are considered which include those where the location and/or scale parameters are equal. Estimable parameters (if any) are characterized in all these cases. Unlike the one sample case, one or the other failure rate is sometimes

estimable in the two sample problem if it corresponds to the population for which the censoring time is larger.

2.2 Unbiased Estimation in the One Sample Problem

Suppose n items, whose lifetimes have pdf (1.1.1), are put to test in an experiment of fixed duration t. Since testing is done with replacement and with all the assumptions this kind of sampling implies, then the joint pdf of failure times and R is given by equation (1.1.5) namely

$$f(x_{(1)},\ldots,x_{(r)},r)=(n\zeta)^r e^{-n\zeta(t-n)} I_{[\eta<x_{(1)}<\ldots<x_{(r)}<t]} \quad r = 1,2\ldots$$

$$P(R = 0) = e^{-n\zeta(t-n)}.$$

Recall that when R = 0, $X_{(1)} > t$, so we define $X_{(1)}$ as the time of the first failure if it is less than t and $X_{(1)} = t$ otherwise.

In addition Wright et al. (1978) have shown via the factorization criterion that $(X_{(1)}, R)$ is sufficient for (η,ζ). The following lemma shows that it is also complete and gives its pdf

Lemma 2.2.1 The statistic $(X_{(1)},R)$ has pdf given by

$$f(x_{(1)},r) = \frac{(n\zeta)^r}{(r-1)!} (t-x_{(1)})^{r-1} e^{-n\zeta(t-\eta)} \quad \eta < x_{(1)}<t \quad r = 1,2,\ldots$$

$$P(R = 0, X_{(1)} = t) = e^{-n\zeta(t-\eta)} \tag{2.2.1}$$

and the family of distributions induced by $X_{(1)}$ and R is complete.

Proof The fact that $(X_{(1)},R)$ has pdf given by (2.2.1) is easily

seen by writing,

$$f(x_{(1)},r) = f(x_{(1)}|r)P(R=r). \quad \text{for } r = 1,2... \tag{2.2.2}$$

so that using Lemma 1.1.2,

$$f(x_{(1)}|r) = \int_{x_{(1)}}^{t} \cdots \int_{x_{(r-1)}}^{t} \frac{r!}{(t-\eta)^r}\, dx_{(r)}\cdots dx_{(2)}$$

$$= \frac{r(t-x_{(1)})^{r-1}}{(t-\eta)^r}, \quad \eta < x_{(1)} < t; \tag{2.2.3}$$

Also, for r=0,

$$P(X_{(1)} = t|R=0) = 1 \tag{2.2.4}$$

Hence, combining (2.2.2) through (2.2.4) and using the fact that $R \sim$ Poisson $(n\zeta(t-\eta))$ (by using Lemma 1.1.1, in Chapter One) we obtain (2.2.1).

To show completeness, let $h(X_{(1)},R)$ be a measurable function of $(X_{(1)},R)$ such that $E_{\eta,\zeta}h(X_{(1)},R) = 0$ for all $\eta\varepsilon(0,t]$, $\zeta>0$. This last statement implies that

$$\sum_{r=1}^{\infty} \int_{\eta}^{t} h(x,r)(n\zeta)^r \frac{(t-x)^r}{(r-1)!}\, dx + h(t,0) = 0 \tag{2.2.5}$$

for all $\eta\varepsilon(0,t]$, $\zeta > 0$.

Since, for a fixed η, (2.2.5) is a power series in ζ then after equating coefficients of ζ^r on both sides one gets

$$h(t,0) = 0$$

$$\int_{\eta}^{t} nh(x,r)\frac{(n(t-x))^{r-1}}{(r-1)!}\, dx = 0, \quad r > 1. \tag{2.2.6}$$

Fix $\eta > 0$ and choose $\eta_o\varepsilon(0,t]$ such that $\eta_o < \eta$. Then

$$\int_{\eta_o}^{t} nh(x,r)\frac{(n(t-x))^{r-1}}{(r-1)!}\, dx = 0. \tag{2.2.7}$$

$$\int_{\eta_o}^{t} nh(x,r)\frac{(n(t-x))^{r-1}}{(r-1)!}\,dx = 0. \tag{2.2.7}$$

Subtracting (2.2.6) from the above expression one gets

$$\int_{\eta_o}^{\eta} h(x,r)n^r \frac{(t-x)^{r-1}}{(r-1)!}\,dx = 0 \tag{2.2.8}$$

for all $0 < \eta_0 < \eta < t$.

Writing $s(x_{(1)},r) = h(x_{(1)},r)\; n^r \dfrac{(t-x_{(1)})^{r-1}}{(r-1)!}$

then

$\int_{\eta_o}^{\eta} s(x,r)dx = 0$ for all $\eta_o,\eta \in (0,t]$

such that $\eta_o < \eta$ and $r > 1$, if and only if

$\int_{\eta_o}^{\eta} s^+(x,r)dx = \int_{\eta_o}^{\eta} s^-(x,r)dx$

The last statement is true if and only if

$\int_B s^+(x,r)dx = \int_B s^-(x,r)dx$ for all $B \in$

where = σ-algebra of Borel sets.

which in turn means

$s^+(x_{(1)},r) = s^-(x_{(1)},r)$ a.e. for $\eta < x_{(1)} < t$, and $r > 1$

i.e.

$s(x_{(1)},r) = 0$ a.e. for $\eta < x_{(1)} < t$ and $r > 1$

which means

$h(x_{(1)},r) = 0$ a.e. for $\eta < x_{(1)} < t$ and $r > 1$.

Hence

$h(x_{(1)},r) = 0$ a.e. for $\eta < x_{(1)} < t$, $r > 0$. □

Note that since $(X_{(1)},R)$ is complete sufficient for (η,ζ), if a parametric function $h(\eta,\zeta)$ is estimable, using the Rao-

Blackwell-Lehmann-Scheffe (RBLS) Theorem, there exists a UMVUE of $h(\eta,\zeta)$ based on $(X_{(1)},R)$.

This shows that if there does not exist any unbiased estimator of $h(\eta,\zeta)$ based on $(X_{(1)},R)$, $h(\eta,\zeta)$ does not have an unbiased estimator.

The first main result of this section is as follows.

Theorem 2.2.1 $h(\eta,\zeta)$ is estimable only if $h(\eta,\zeta)$ is of the form $h(\eta,\zeta) = \Sigma_{r=0}^{\infty} u_r(\eta)\zeta^r$, where $u_0(\eta)$ does not depend on η and $u_r(t) = 0$ for $r\geq 1$.

Proof Suppose $h(\eta,\zeta)$ is estimable. Then there exists some statistic $g(X_{(1)},R)$ such that

$$E_{\eta,\zeta}g(X_{(1)},R) = h(\eta,\zeta) \text{ for all } \eta < t, \text{ and } \zeta > 0. \quad (2.2.9)$$

This means via Lemma 2.2.1 that

$$h(\eta,\zeta) = \Sigma_{r=1}^{\infty}\int_{\eta}^{t} g(x,r) \exp[-n\zeta(t-\eta)](n\zeta)^r(t-x)^{r-1}/(r-1)!\, dx$$

$$+g(t,0) \exp[-n\zeta(t-\eta)], \quad (2.2.10)$$

for all $\eta < t$ and $\zeta > 0$.

Note that for every fixed η, the right hand side of (2.2.10) is a power series in ζ so that the left hand side of (2.2.10) must also be a power series in ζ. Thus $h(\eta,\zeta)$ must be of the form $\Sigma_{r=0}^{\infty} u_r(\eta)\zeta^r$. Equating now the coefficient of ζ^0 on both sides of (2.2.10), one gets $u_0(\eta) = g(t,0)$ for all $\eta(<t)$ which shows that $u_0(\eta)$ does not depend on η. Also $\sum_{r=0}^{\infty} u_r(t)\zeta^r = g(t,0)$ for all $\zeta > 0$ which implies that $u_r(t) = 0$ for all $r \geq 1$ and $u_0(t) = g(t,0)$. □

Note that we are tacitly assuming the convergence of the power series $\sum_{r=0}^{\infty} u_r(\eta)\zeta^r$ for all $\zeta \in (0,\infty)$. In what follows, we assume all power series encountered to be convergent over the range of possible values of the failure rates.

Remark 1 It follows as a consequence of Theorem 2.2.1 that a function $u(\eta)$ is not estimable unless $u(\eta)$ is a constant. In particular, there does not exist any unbiased estimator of the location parameter η. Also, the scale parameter ζ^{-1} is not estimable, because it cannot be expressed as a power series in ζ.

Thus, we have narrowed down the entire class of parametric functions $h(\eta,\zeta)$ to parametric functions of the form $\Sigma_{r=0}^{\infty} u_r(\eta)\zeta^r$ (where $u_0(\eta)$ does not depend on η) as potential candidates for being estimable. The next theorem characterizes all estimable functions within this class, where $u_r(\eta)$ is differentiable in η.

Theorem 2.2.2 *Suppose* $u_r(x)$ *is differentiable in* $x < t$. *Then* $\Sigma_{r=0}^{\infty} u_r(\eta)\zeta^r$ *admits an unbiased estimator based on* $X_{(1)}$ *and* R *if and only if* $u_0(\eta)$ *does not depend on* η *and* $u_r(t) = 0$ *for all* $r \geq 1$. *Furthermore, the class of unbiased estimators of differentiable (in* η*) parametric functions, consists exactly of functions of* x *and* r *which are continuous in* x *for* $x \in (0,t)$, $r \geq 1$.

Proof Necessity has been established on the previous theorem. To prove sufficiency assume $g(x,r)$ is a continous function of $x\varepsilon(0,t)$ for $r \geq 1$. Then, using (2.2.1) one gets

$$\Sigma_{r=0}^{\infty} u_r(\eta)\zeta^r = \Sigma_{r=1}^{\infty}\int_{\eta}^{t} g(x,r)(n\zeta)^r \exp\left(-n\zeta(t-\eta)\right)(t-x)^{r-1}/(r-1)!dx$$

$$+g(t,0)\exp\big(-n\zeta(t-\eta)\big) \tag{2.2.11}$$

for all $\eta < t$, $\zeta > 0$. This implies that

$$\big(\Sigma_{r=0}^{\infty}u_r(\eta)\zeta^r\big)\big(\Sigma_{r=0}^{\infty}\big(n\zeta(t-\eta)\big)^r/r!\big)$$

$$= \Sigma_{r=1}^{\infty}n^r\zeta^r\int_{\eta}^{t}g(x,r)(t-x)^{r-1}/(r-1)!\ dx + g(t,0), \tag{2.2.12}$$

for all $\eta < t, \zeta > 0$.

Equating the coefficients of $\zeta^r (r \geq 1)$ on both sides of (2.2.12), one gets

$$\begin{aligned} v_r(\eta) &= \Sigma_{j=0}^{r}u_j(\eta)\big(n(t-\eta)\big)^{r-j}/(r-j)! \\ &= \int_{\eta}^{t}g(x,r)n^r(t-x)^{r-1}/(r-1)!dx, \end{aligned} \tag{2.2.13}$$

for all $\eta < t$. Note that $v_0(\eta) = u_0(\eta) = g(t,0)$ for all $\eta < t$. Now differentiating both sides of (2.2.13) with respect to η, and then writing x for η, one gets

$$g(x,r) = -n^{-r}(t-x)^{r-1}(r-1)!\ v_r'(x),\ x < t,\ r \geq 1. \tag{2.2.14}$$

Using (2.2.14) and the fact that $v_r(t) = u_r(t) = 0$ for $r \geq 1$ and $v_0(t) = u_0(t) = g(t,0)$, one gets

$$\begin{aligned} E[g(X_{(1)},R)] &= \big[-\Sigma_{r=1}^{\infty}\zeta^r\int_{\eta}^{t}v_r'(x)dx + g(t,0)\big]\ \exp\big(-n\zeta(t-\eta)\big) \\ &= \big[\Sigma_{r=1}^{\infty}\zeta^r\big(v_r(\eta) - v_r(t)\big) + g(t,0)\big]\ \exp\big(-n\zeta(t-\eta)\big) \\ &= \big[\Sigma_{r=0}^{\infty}\zeta^r\big(v_r(\eta) - v_r(t)\big) + g(t,0)\big]\ \exp\big(-n\zeta(t-\eta)\big) \\ &= \{\Sigma_{r=0}^{\infty}\zeta^r\Sigma_{j=0}^{r}u_j(\eta)\big(n(t-\eta)\big)^{r-j}/(r-j)!\}\exp\big(-n\zeta(t-\eta)\big) \\ &= \Sigma_{j=0}^{\infty}u_j(\eta)\zeta^j. \quad \square \end{aligned} \tag{2.2.15}$$

Note that since an estimator has been constructed using a complete sufficient statistic, it follows via RBLS that it has minimum variance in the class of all unbiased estimators of this parametric function.

Next, let $g_0(x,r)$ be a possibly discontinuous (in x) function, whose expectation $h_0(\eta,\zeta)$ say, is differentiable in η. By Theorem 2.2.1 and the first part of Theorem 2.2.2 it follows that $h_0(\eta,\zeta)$ must be of the form $\Sigma^{\infty}_{r=0}u_r(\eta)\zeta^r$ where $u_0(\eta)$ does not depend on η and $u_r(t) = 0$ for $r \geq 1$. But for such a parametric function by the second part of Theorem 2.2.2, we can construct an unbiased estimator g(X,R) which belongs to the class of functions of x and r which are continuous in $0 < x < t$ for $r \geq 1$. This implies that $E_{\eta,\zeta}\left(g_0(X,R) - g(X,R)\right) = 0$ all $\eta\varepsilon(0,t]$, $\zeta > 0$ which in turn means that $g_0(X,R) = g(X,R)$ with probability one, by completeness of $X_{(1)}$ and R.

Hence, only functions of $X_{(1)}$ and R which are continuous in $x_{(1)}$ for $\eta < x_{(1)} < t$, and $r \geq 1$ can be unbiased estimators of parametric functions $h(\eta,\zeta)$ which are differentiable in η.

Remark 2 It follows from Theorem 2.2.2 that the failure rate ζ itself is not estimable. Indeed, any non-constant power series $k(\zeta)$ of ζ is not estimable where the coefficients in the power series do not involve η. However, the function $\exp\{-n\zeta(t-\eta)\}$ is estimable. Indeed the function $I_{[R=0]}$ is the UMVUE of this estimable function.

Remark 3 In the case when η is known, it can be seen from (1.1.5) and the factorization criterion that R is sufficient for ζ. Since $R \sim$ Poisson $(n\zeta(t-\eta))$ it is well known that it has a complete family of pdfs.

The following lemma characterizes the class of estimable functions in the RBLS sense for this situation.

Lemma 2.2.2 If $R \sim$ Poisson $(n\zeta c)$, where c is a known constant, then a parametric function $K(\zeta)$ is estimable if and only if it is a power series in ζ.

Proof Let $g(R)$ be an unbiased estimator for $K(\zeta)$. This implies that

$$K(\zeta) = \Sigma_{r=0}^{\infty} g(r) \frac{(nc)^r}{r!} \zeta^r e^{-n\zeta c}, \quad 0 < \zeta < \infty. \tag{2.2.16}$$

Since the right hand side of (2.2.16) is a power series in ζ, we must have

$$K(\zeta) = \Sigma_{r=0}^{\infty} a_r \zeta^r \quad 0 < \zeta < \infty. \tag{2.2.17}$$

To prove sufficiency write, from (2.2.16) and (2.2.17)

$$\Sigma_{r=0}^{\infty} a_r \zeta^r = \Sigma_{r=0}^{\infty} g(r) \frac{(nc)^r}{r!} \zeta^r e^{-n\zeta c}$$

This implies that

$$\Sigma_{r=0}^{\infty} \zeta^r \Sigma_{j=0}^{r} a_j \frac{(nc)^{r-j}}{(r-j)!} = \Sigma_{j=0}^{\infty} \zeta^r g(r) \frac{(nc)^r}{r!}. \tag{2.2.18}$$

Equating coefficients on both sides we obtain

$$g(r) = \frac{r!}{(nc)^r} \Sigma_{j=0}^{r} a_j \frac{(nc)^{r-j}}{(r-j)!} \qquad \text{for } r \geq 0. \tag{2.2.18a}$$

Hence,

$$Eg(R) = \Sigma_{r=0}^{\infty} \left[\frac{r!}{(nc)^r} \Sigma_{j=0}^{r} a_j \frac{(nc)^{r-j}}{(r-j)!}\right] \frac{(n\zeta c)^r}{r!} e^{-n\zeta c}$$

$$= \Sigma_{r=0}^{\infty} \zeta^r \Sigma_{j=0}^{r} a_j \frac{(nc)^{r-j}}{(r-j)!} e^{-n\zeta c}$$

$$= (\Sigma_{r=0}^{\infty} a_r \zeta^r)(\Sigma_{r=0}^{\infty} \frac{(nc)^r}{r!} \zeta^r) e^{-n\zeta c}$$

$$= \Sigma_{r=0}^{\infty} a_r \zeta^r .$$

Remark 4 Suppose now ζ is known, but η is unknown. In this case $X_{(1)}$ is sufficient for η, and has pdf given by

$$f(x_{(1)}) = n\zeta \exp[-n\zeta(x_{(1)}-\eta)] \quad \text{if } \eta < x_{(1)} < t. \qquad (2.2.19)$$

$$P(X_{(1)} = t) = \exp(-n\zeta(t-\eta)) \qquad (2.2.20)$$

Theorem 2.2.3 $X_{(1)}$ has a complete family of distributions.

Proof Let $g(X_{(1)})$ be a measurable function of $X_{(1)}$ such that $E_\eta g(X_{(1)}) = 0$. Using (2.2.19)-(2.2.20) if follows that

$$q_1(\eta) = \int_\eta^t n\zeta \exp[-n\zeta x] g(x)dx + g(t)\exp[-n\zeta t] = 0 \qquad (2.2.21)$$

Chose $\eta' \varepsilon (0,t_1)$ such that $\eta' < \eta$, then

$$q_1(\eta') - q_1(\eta) = \int_{\eta'}^{\eta} g(x) n\zeta \exp(-n\zeta x)dx = 0$$

for all $0 < \eta' < \eta \leq t$, which follows, if and only if

$$\int_{\eta'}^{\eta} g^+(x) n\zeta \exp(-n\zeta x)dx = \int_{\eta'}^{\eta} g^-(x) n\zeta \exp(-n\zeta x)dx \qquad (2.2.22)$$

Put $q_2(x) = g(x)\, n\zeta \exp(-n\zeta x)$. Then (2.2.22) implies that

$$\int_B q_2^+(x_{(1)})dx = \int_B q_2^-(x)dx \text{ for all } B \varepsilon \mathfrak{B} \qquad (2.2.23)$$

where $\mathfrak{B}$ = σ-algebra of Borel sets.

Equation (2.2.23) is true if and only if

$$q_2^+(x_{(1)}) = q_2^-(x_{(1)}) \text{ for almost all } x_{(1)} \varepsilon (\eta,t)$$

i.e. if and only if

$$q_2(x_{(1)}) = 0 \quad \text{for almost all } x_{(1)} \varepsilon (\eta,t)$$

which in turn means

$$g(x_{(1)}) = 0 \text{ for almost all } x_{(1)} \varepsilon (\eta,t). \qquad (2.2.24)$$

Using (2.2.24) and going back to (2.2.21) it follows that

$$g(x_{(1)}) = 0 \text{ for almost all } \eta < x_{(1)} \leq t.$$

In this case, any differentiable function $u(\eta)$ admits an unbiased estimator $g(X_{(1)})$ if and only if $u(t) = g(t)$. In such case $g(x_{(1)})$ is necessarily continuous for $\eta < x_{(1)} < t$.

To see this note that if $g(X_{(1)})$ is an unbiased estimator of $u(\eta)$, where $u(\eta)$ is differentiable in η, one has

$$u(\eta) = \int_\eta^t g(x)n\zeta\exp(-n\zeta(x-\eta))dx + g(t)\exp(-n(t-\eta)). \qquad (2.2.25)$$

which implies $u(t) = g(t)$.

Next, assume $g(x_{(1)})$ is continous in $x_{(1)}$ for $\eta < x_{(1)} < t$, and $g(t) = u(t)$. Then

$$u(\eta)\exp(-n\zeta\eta) = \int_\eta^t g(x)n\zeta\exp(-n\zeta x)dx + g(t)\exp(-n\zeta t). \quad (2.2.26)$$

Differentiating both sides of (2.2.26) with respect to $\eta < t$, one gets

$$(u'(\eta)-n\zeta u(\eta))\exp(-n\zeta\eta) = -g(\eta)n\zeta\exp(-n\zeta\eta). \qquad (2.2.27)$$

From (2.2.27), one gets

$$g(x_{(1)}) = u(x_{(1)}) - u'(x_{(1)})(n\zeta)^{-1} \text{ if } \eta < x_{(1)} < t. \quad \square$$

Also, using the fact that, $g(t) = u(t)$, it follows that $g(X_{(1)})$ is the UMVUE of $u(\eta)$. In particular η has the UMVUE $X_{(1)} - (n\zeta)^{-1} I_{[\eta < X_{(1)} < t]}$.

It also follows, using familiar arguments, that only functions that are continuous in x for $\eta < x < t$ qualify as unbiased estimators of $u(\eta)$, where $u(\eta)$ is differentiable.

Note in (2.2.25) that if $u(\eta)$ is estimable but not necessarily differentiable we must still have $u(t) = g(t)$.

2.3 Unbiased Estimation in the Two Sample Problem

Suppose that two independent sets of items are put to test, where the first set contains n_1 elements, while the second set contains n_2 elements. As in Section 2.2, a failed item is replaced instantaneously by a similar item. Recall from Chapter One that the lifetimes for items in set i are assumed to be iid exponential with location parameter η_i, and failure rate ζ_i (i=1,2), while replacement items, on the ith group, have independent lifetimes with location parameter zero and failure rate ζ_i. The censoring times for these two sets are denoted by t_1 and t_2. Assume that $\eta_i < t_i$ (i=1,2). We denote by R_i the number of failures before time t_i for the set i (i=1,2). Then from Lemma 1.1.1, Chapter One, R_i's are independent with $R_i \sim$ Poisson $(n_i\zeta_i(t_i - \eta_i))$, i=1,2.

Given $R_i = r_i$ (> 0), the ordered failure times for the set i say $X_{(i1)} < X_{(i2)} < \cdots < X_{(ir_i)}$ are the ordered values of a random sample of size r_i from the uniform (η_i, t_i) distribution, also define $X_{(i1)} = t_i$ if $R_i = 0$ (i=1,2).

Several cases need to be considered. First consider the case when η_1, η_2, ζ_1 and ζ_2 are all distinct and unknown. In this case, the joint pdf of all the observations is given by (1.1.9), in Chapter One, and by the factorization criterion $(X_{(11)}, X_{(21)}, R_1, R_2)$ is sufficient for $(\eta_1, \eta_2, \zeta_1, \zeta_2)$. Their joint pdf is given by

$f(x_{(11)},x_{(21)},r_1,r_2)$

$$= \prod_{i=1}^{2} \frac{(\eta_i\zeta_i)^{r_i}}{(r_i-1)!} (t_i-x_{(i1)})^{r_i-1} e^{-\eta_i\zeta_i(t_i-\eta_i)} I_{[\eta_i<x_{(i1)}<t_i]},$$

for $r_1>0$, $r_2>0$

$$= e^{-\sum_{i=1}^{2}\eta_i\zeta_i(t_i\eta_i)} \frac{(\eta_2\zeta_2)^{r_2}}{(r_2-1)!} (t_2-x_{(21)})^{r_2-1} I_{[\eta_2<x_{(21)}<t_2]}$$

for $r_1= 0$, $r_2 > 0$

$$= e^{-\sum_{i=1}^{2} \eta_i\zeta_i(t_i-\eta_i)} \frac{(\eta_1\zeta_1)^{r_1}}{(r_1-1)!} (t_1-x_{(11)})^{r_1-1} I_{[\eta_1< x_{(11)}< t_1]}$$

for $r_1 > 0$, $r_2 = 0$

$$= e^{-\sum_{i=1}^{2} \eta_i\zeta_i(t_i-\eta_i)} \qquad r_1 = 0,\ r_2 = 0 \qquad (2.3.1)$$

<u>Theorem 2.3.1</u> $(X_{(11)},X_{(21)}, R_1, R_2)$ has a complete family of distributions.

<u>Proof</u> Let $h(X_{(11)},X_{(21)}, R_1, R_2)$ be a measurable function of $(X_{(11)},X_{(21)}, R_1, R_2)$ such that $E_{\eta_1\eta_2\zeta_1\zeta_2} h(X_{(11)},X_{(21)}, R_1, R_2)=0$ for all $\eta_1\varepsilon(0,t_1]$, $\eta_2\varepsilon(0,t_2]$, $\zeta_1 > 0$ and $\zeta_2 > 0$.

Then

$$E_{\eta_1\eta_2\zeta_1\zeta_2} h(X_{(11)},X_{(21)}, R_1, R_2) =$$

$$E_{\eta_2\zeta_2}E_{\eta_1\zeta_1}[h(X_{(11)},X_{(21)}, R_1, R_2)|X_{(21)},R_2] = 0$$

By independence $E_{\eta_1\zeta_1} (h(X_{(11)},X_{(21)}, R_1, R_2) |x_{(21)},r_2)$ $= E_{\eta_1\zeta_1} h(X_{(11)},x_{(21)}, R_1, r_2)$ and $E_{\eta_1\zeta_1}h(X_{(11)},X_{(21)}, R_1, R_2) =$

$g(\eta_1, \zeta_1, X_{(21)}, R_2)$ (say) is not a function of η_2 or ζ_2 since the parameters are all distinct i.e., when integrating with respect to $(X_{(11)}, R_1)$

$$g(\eta_1,\zeta_1,X_{(21)},R_2) = E_{\eta_1,\zeta_1}(h(X_{(11)},X_{(21)}, R_1, R_2))$$

is still a statistic, i.e. it does not depend on any unknown parameters of the distribution of $(X_{(21)}, R_2)$. Hence,

$$E_{\eta_1\eta_2\zeta_1\zeta_2}h(X_{(11)},X_{(21)},R_1,R_2)$$
$$= E_{\eta_2\zeta_2}E_{\eta_1\zeta_1}(h(X_{(11)},X_{(21)},R_1,R_2) = E_{\eta_2\zeta_2}g(\eta_1,\zeta_1,X_{(21)},R_2) = 0$$

By completeness of $(X_{(21)},R_2)$ this implies

$$g(\eta_1,\zeta_1,X_{(21)},R_2)=E_{\eta_1\zeta_1}h(X_{(11)},X_{(21)},R_1,R_2) = 0$$

for all $\eta_2 \varepsilon (0,t_2]$, $\zeta_2 > 0$.

Then for all $\eta_1 \varepsilon (0,t_1]$, $\zeta_1 > 0$ and by completeness of $(X_{(11)},R_1)$,

$$h(X_{(11)},X_{(21)}, R_1, R_2) = 0 \text{ with probability 1.} \qquad \square$$

Next by straightforward generalization of Theorem 2.2.1 we obtain the following result.

<u>Theorem 2.3.2</u> <u>A parametric function</u> $h(\eta_1,\eta_2,\zeta_1,\zeta_2)$ <u>is estimable only if it is of the form</u> $\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} u_{r_1,r_2}(\eta_1,\eta_2)\zeta_1^{r_1}\zeta_2^{r_2}$

<u>where</u> $u_{0,0}(\eta_1,\eta_2)$ <u>does not depend on</u> η_1 <u>nor</u> η_2, $u_{0,r_2}(\eta_1,\eta_2)$ <u>does not depend on</u> η_1 <u>for</u> $r_2 > 1$, $u_{r_1,0}(\eta_1,\eta_2)$ <u>does not depend on</u> η_2 <u>for</u> $r_1 > 1$ <u>and</u> $u_{r_1,r_2}(t_1,t_2) = 0$ for $(r_1,r_2) \neq (0,0)$, $u_{r_1,0}(t_1,\eta_2) = 0$ for $r_1 > 1$, $u_{0,r_2}(\eta_1,t_2) = 0$ for $r_2 > 1$ <u>and all</u> $\eta_1 \varepsilon (0,t_1]$, $\eta_2 \varepsilon (0,t_2]$.

<u>Proof</u> Suppose $h(\eta_1,\eta_2,\zeta_1,\zeta_2)$ is estimable. This in turn implies that there is some statistic $g(X_{(11)},X_{(21)}, R_1, R_2)$ such that

$$E_{\eta_1,\eta_2,\zeta_1,\zeta_2} g(X_{(11)},X_{(21)},R_1,R_2) = h(\eta_1,\eta_2,\zeta_1,\zeta_2) \qquad (2.3.2)$$

for all $0 < \eta_1 < t_1$, $0 < \eta_2 < t_2$, $\zeta_1 > 0$, $\zeta_2 > 0$.

Using equation (2.3.1) in combination with (2.3.2) and writing $U_1 = X_{(11)}$ and $U_2 = X_{(21)}$, one gets

$h(\eta_1,\eta_2,\zeta_1,\zeta_2) =$

$$\sum_{r_2=1}^{\infty}\sum_{r_1=1}^{\infty}\int_{\eta_1}^{t_1}\int_{\eta_2}^{t_2} g(u_1,u_2,r_1,r_2)\frac{(n_1\zeta_1)^{r_1}}{(r_1-1)!}\frac{(n_2\zeta_2)^{r_2}}{(r_2-1)!}$$

$$\cdot (t_1-u_1)^{r_1-1}(t_2-u_2)^{r_2-1}\exp\Big[\sum_{i=1}^{2} - n_i\zeta_i(t_i-\eta_i)\Big]du_2du_1$$

$$+ \sum_{r_1=1}^{\infty}\int_{\eta_1}^{t_1} g(u_1,t_2,r_1,0)\frac{(n_1\zeta_1)^{r_1}}{(r_1-1)!}(t_1-u_1)^{r_1-1}$$

$$\cdot\exp\Big[\sum_{i=1}^{2} - n_i\zeta_i(t_i-\eta_i)\Big]du_1$$

$$+ \sum_{r_2=1}^{\infty}\int_{\eta_2}^{t_2} g(t_1,u_2,0,r_2)\frac{(n_2\zeta_2)^{r_2}}{(r_2-1)!}(t_2-u_2)^{r_2-1}$$

$$\cdot\exp\Big[\sum_{i=1}^{2} - n_i\zeta_i(t_i-\eta_i)\Big]du_2$$

$$+ g(t_1,t_2,0,0)\exp\Big[-\sum_{i=1}^{2} n_i\zeta_i(t_i-\eta_i)\Big] \qquad (2.3.3)$$

for all $0 < \eta_1 < t_1$, $0 < \eta_2 < t_2$, $\zeta_1 > 0$, $\zeta_2 > 0$.

Note that for every fixed η_1 and η_2, the right hand side of (2.3.3) is a bivariate power series in ζ_1 and ζ_2. Therefore $h(\eta_1,\eta_2,\zeta_1,\zeta_2)$ must be of the form

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} u_{r_1,r_2}(\eta_1,\eta_2)\zeta_1^{r_1}\zeta_2^{r_2}. \qquad (2.3.4)$$

Equating the coefficients for $\zeta_1^0\ \zeta_2^0$ on both sides one gets $u_{0,0}(\eta_1,\eta_2) = g(t_1,t_2,0,0)$ for all $\eta_1 \varepsilon (0,t_1]$, $\eta_2 \varepsilon (0,t_2]$ which implies that $u_{0,0}(\eta_1,\eta_2)$ does not depend on η_1 nor η_2

and $\sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} u_{r_1,r_2}(t_1,t_2)\zeta_1^{r_1}\zeta_2^{r_2} = g(t_1,t_2,0,0)$ for all $\zeta_1 > 0$,

$\zeta_2 > 0$ which implies that

$u_{r_1,r_2}(t_1,t_2) = (0,0)$ for $(r_1,r_2) \neq (0,0)$

and $u_{0,0}(t_1,t_2) = g(t_1,t_2,0,0)$.

Note also that equating the coefficients of $\zeta_1^0\zeta_2^{r_2}$, $r_2 > 0$

we get

$$u_{0,r_2}(\eta_1,\eta_2) = g(t_1,t_2,0,0)\,\frac{(\eta_2(t_2-\eta_2))^{r_2}}{r_2!}$$

$$+ \sum_{r_2=1}^{\infty} \sum_{j_2=0}^{r_2} \frac{(\eta_2(t_2-\eta_2))^{j_2}}{j_2!} \int_{\eta_2}^{t_2} g(t_1,u_2,0,r_2-j_2)$$

$$\cdot \frac{\eta_2^{r_2-j_2}}{(r_2-j_2-1)!}(t_2-u_2)^{r_2-j_2-1} du_2, \text{ for } \eta_1 \varepsilon (0,t_1], \eta_2 \varepsilon (0,t_2]$$

which shows that $u_{0,r_2}(\eta_1,\eta_2)$ does not depend on η_1

and $u_{0,r_2}(\eta_1,t_2) = g(t_1,t_2,0,0)$ $\quad r_2 = 0$

$= 0$ $\quad r_2 \geq 1$

Likewise for $r_1 > 0$, equating the coefficients of $\zeta_1^{r_1}\zeta_2^0$, we obtain

$$u_{r_1,0}(\eta_1,\eta_2) = g(t_1,t_2,0,0)\,\frac{(\eta_1(t_1-\eta_1))^{r_1}}{r_1!}$$

$$+ \sum_{r_1=1}^{\infty} \sum_{j_1=0}^{r_1} \frac{(\eta_1(t_1-\eta_1))^{j_1}}{j_1!} \int_{\eta_1}^{t_1} g(u_1,t_2,r_1-j_1,0)\,\frac{\eta_1^{r_1-j_1}}{(r_1-j_1-1)!}$$

$$\cdot\ (t_1-u_1)^{r_1-j_1-1}\ du_1$$

for $\eta_1 \varepsilon (0,t_1]$, $\eta_2 \varepsilon (0,t_2]$ which shows that $u_{r_1,0}(\eta_1,\eta_2)$ does not depend on η_2 for $r_1 > 1$

$$\text{and } u_{r_1,0}(t_1,\eta_2) = g(t_1,t_2,0,0) \qquad r_1 = 0$$

$$= 0 \qquad r_1 > 1 \qquad \square$$

Accordingly, any non-trivial parametric function involving η_1 and η_2 only is not estimable. Also, a function $K(\zeta_1,\zeta_2)$ is estimable only if it is a bivariate power series in ζ_1 and ζ_2. Thus, the scale parameters ζ_1^{-1} and ζ_2^{-1} are not estimable.

Our next result generalizes Theorem 2.2.2 to cover our present case.

<u>Theorem 2.3.3</u> <u>A parametric function</u> $\sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} u_{r_1,r_2}(\eta_1,\eta_2)\zeta_1^{r_1}\zeta_2^{r_2}$ <u>where</u> $u_{0,0}(\eta_1,\eta_2)$ <u>does not depend on</u> η_1 and η_2, $u_{0,r_2}(\eta_1,\eta_2)$ <u>does not depend on</u> η_1 <u>for</u> $r_2 > 1$, <u>and</u> $u_{r_1,0}(\eta_1,\eta_2)$ <u>does not depend on</u> η_2 <u>for</u> $r_1 > 1$ <u>and which is differentiable in</u> η_1 <u>and</u> η_2, <u>admits an unbiased estimator if and only if</u> $u_{r_1,r_2}(t_1,t_2) = 0$ <u>for</u> $(r_1, r_2) \neq (0,0)$, $u_{r_1,0}(t_1,\eta_2) = 0$ for $r_1 > 1$, <u>and</u> $u_{0,r_2}(\eta_1,t_2) = 0$ <u>for</u> $r_2 > 1$ <u>for all</u> $0 < \eta_1 < t_1$ and $0 < \eta_2 < t_2$.

<u>Proof</u> Necessity has been established in Theorem 2.3.2.

To prove sufficiency assume $g(u_1,u_2,r_1,r_2)$ is continuous in $\eta_1 < u_1 < t_1$, $\eta_2 < u_2 < t_2$ for $(r_1,r_2) \neq (0,0)$. Then using (2.3.3) and (2.3.4) one gets

$$\left[\sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} u_{r_1,r_2}(\eta_1, \eta_2)\zeta_1^{r_1}\zeta_2^{r_2}\right]$$

$$\cdot\left[\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty}\frac{(n_1(t_1-\eta_1))^{r_1}}{r_1!}\frac{(n_2(t_2-\eta_2))^{r_2}}{r_2!}\zeta_1^{r_1}\zeta_2^{r_2}\right]$$

$$= \sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\int_{\eta_1}^{t_1}\int_{\eta_2}^{t_2} g(u_1,u_2,r_1,r_2)\frac{(n_1\zeta_1)^{r_1}}{(r_1-1)!}\frac{(n_2\zeta_2)^{r_2}}{(r_2-1)!}$$

$$\cdot (t_1-u_1)^{r_1-1}(t_2-u_2)^{r_2-1}du_2du_1$$

$$+\sum_{r_1=1}^{\infty}\int_{\eta_1}^{t_1} g(u_1,t_1,r_1,0)\frac{(n_1\zeta_1)^{r_1}}{(r_1-1)!}(t_1-u_1)^{r_1-1}du_1$$

$$+\sum_{r_2=1}^{\infty}\int_{\eta_2}^{t_2} g(t_1,u_2,0,r_2)\frac{(n_2\zeta_2)^{r_2}}{(r_2-1)!}(t_2-u_2)^{r_2-1}du_2$$

$$+ g(t_1,t_2,0,0)$$

for all $\eta_1\varepsilon(0,t_1]$, $\eta_2\varepsilon(0,t_2]$, $\zeta_1 > 0$, $\zeta_2 > 0$. (2.3.5)

Equating the coefficients of $\zeta_1^{r_1}\zeta_2^{r_2}$ on both sides of (2.3.5), one obtains that for $u < \eta_1 < t_1$, $0 < \eta_2 < t_2$, $r_1 \geq 1$, $r_2 \geq 1$

$$v_{r_1,r_2}(\eta_1,\eta_2) = \sum_{j_1=0}^{r_1}\sum_{j_2=0}^{r_2} u_{j_1,j_2}(\eta_1,\eta_2)\frac{(n_1(t_1-\eta_1))^{r_1-j_1}}{(r_1-j_1)!}\cdot\frac{(n_2(t_2-\eta_2))^{r_2-j_2}}{(r_2-j_2)!}$$

$$= \int_{\eta_1}^{t_1}\int_{\eta_2}^{t_2} g(u_1,u_2,r_1,r_2)\frac{n_1^{r_1}}{(r_1-1)!}\frac{n_2^{r_2}}{(r_2-1)!}$$

$$\cdot (t_1-u_1)^{r_1-1}(t_2-u_1)^{r_2-1}du_2du_1 \qquad r_1\geq 1, r_2\geq 1; \quad (2.3.6)$$

Note that in (2.3.6) when at least one $\eta_i = t_i$, $v_{r_1,r_2}(\eta_1,\eta_2) = 0$

Also, for $0 < \eta_2 < t_2$, $r_1 = 0$, $r_2 \geq 1$

$$v_{0,r_2}(\eta_1,\eta_2) = \sum_{j_2=0}^{r_2} u_{0,j_2}(\eta_1,\eta_2) \frac{(n_2(t_2-\eta_2))^{r_2-j_2}}{(r_2-j_2)!}$$

$$= \int_{\eta_2}^{t_2} g(t_1,u_2,0,r_2) \frac{(n_2\zeta_2)^{r_2}}{(r_2-1)!} (t_2-u_2)^{r_2-1} du_2 \qquad (2.3.7)$$

And for $0 < \eta_1 < t_1$, $r_1 > 1$, $r_2 = 0$.

$$v_{r_1,0}(\eta_1,\eta_2) = \sum_{j_1=0}^{r_1} u_{j_1,0}(\eta_1,\eta_2) \frac{(n_1(t_1-\eta_1))^{r_1-j_1}}{(r_1-j_1)!}$$

$$= \int_{\eta_1}^{t_1} g(u_1,t_2,r_1,0) \frac{(n_1\zeta_1)^{r_1}}{(r_1-1)!} (t_1-u_1)^{r_1-1} du_1 \qquad (2.3.8)$$

$$v_{0,0}(\eta_1,\eta_2) = u_{0,0}(\eta_1,\eta_2) = g(t_1,t_2,0,0,) \qquad (2.3.9)$$

for all $0 < \eta_1 < t_1$, $0 < \eta_2 < t_2$.

Then, for $r_1 > 1$, $r_2 > 1$, differentiating (2.3.6) with respect to both $\eta_1 < t_1$ and $\eta_2 < t_2$ and writing u_1 for η_1 and u_2 for η_2 one gets

$$g(u_1,u_2,r_1,r_2) = \frac{(r_1-1)!}{n_1^{r_1} n_2^{r_2} (t_1-u_1)^{r_2-1}} \frac{(r_2-1)!}{(t_2-u_2)^{r_2-1}}$$

$$\cdot \frac{\partial^2 v_{r_1,r_2}(u_1,u_2)}{\partial u_1,\partial u_2} \qquad (2.3.10)$$

Likewise from (2.3.7) and for $r_1 = 0$, $r_2 > 1$ and $\eta_2 < t_2$

$$g(t_1,u_2 0,r_2) = -\frac{(r_2-1)!}{n_2^{r_2}(t_2-u_2)^{r_2-1}} \frac{v_{0,r_2}(t_1,u_2)}{\partial u_2} \qquad (2.3.11)$$

In (2.3.11) we have used the fact that $v_{0,r_2}(\eta_1,\eta_2)$ does not

depend on η_1. From (2.3.8) and for $r_1 \geq 1$, $r_2 = 0$ and $\eta_1 < t_1$

$$g(u_1,t_2,r_1,0) = -\frac{(r_1-1)!}{n_1^{r_1}(t_1-u_1)^{r_1-1}} \frac{\partial v_{r_1,0}(u_1,t_2)}{\partial u_1} \tag{2.3.12}$$

Here, in (2.3.12), we have used the fact that $v_{r_1,0}(\eta_1,\eta_2)$ does not depend on η_2.

Define $b = \sum_{i=1}^{2} n_i\zeta_i(t_i-\eta_i)$.

Then, using (2.3.9) through (2.3.12) and the fact that

$v_{0,0}(t_1,t_2) = u_{0,0}(t_1,t_2)$, $v_{r_1,r_2}(\eta_1,t_2) = 0$ for $r_1 > 0$,

$r_2 \geq 1$ and $v_{r_1,r_2}(t_1,\eta_2) = 0$ for $r_1 \geq 1$, $r_2 > 0$

we obtain

$Eg(U_1,U_2,R_1,R_2) =$

$$\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\int_{\eta_1}^{t_1}\int_{\eta_2}^{t_2} \frac{e^{-b}(r_1-1)!\ (r_2-1)!}{n_1^{r_1}\, n_2^{r_2}\,(t_1-u_1)^{r_1-1}\,(t_2-u_2)^{r_2-1}}$$

$$\cdot \frac{\partial^2 v_{r_1,r_2}(u_1,u_2)}{\partial u_1\,\partial u_2} \frac{(n_1\zeta_1)^{r_1}(t_1-u_1)^{r_1-1}}{(r_1-1)!} \frac{(n_2\zeta_2)^{r_2}(t_2-u_2)^{r_2-1}}{(r_2-1)!}\, du_1 du_2$$

$$-\sum_{r_1=1}^{\infty}\int_{\eta_1}^{t_1} \frac{e^{-b}(r_1-1)!}{n_1^{r_1}(t_1-u_1)^{r_1-1}} \frac{\partial v_{r_1,0}(u_1,t_2)}{\partial u_1} \frac{(n_1\zeta_1)^{r_1}}{(r_1-1)!}(t_1-u_1)^{r_1-1}\, du_1$$

$$-\sum_{r_2=1}^{\infty}\int_{\eta_2}^{t_2} \frac{e^{-b}(r_2-1)!}{n_2^{r_2}(t_2-u_2)^{r_2-1}} \frac{\partial v_{0,r_2}(t_1,u_2)}{\partial u_2} \frac{(n_2\zeta_2)^{r_2}}{(r_2-1)!}(t_2-u_2)^{r_2-1}\, du_2$$

$$+ g(t_1,t_2,0,0)e^{-b}$$

$$= [\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\zeta_1^{r_1}\zeta_2^{r_2}\int_{\eta_1}^{t_1}\int_{\eta_2}^{t_2}\frac{\partial^2 v_{r_1,r_2}(u_1,u_2)}{\partial u_1\partial u_2}du_2,du_1$$

$$-\sum_{r_1=1}^{\infty}\int_{\eta_1}^{t_1}\zeta_1^{r_1}\frac{\partial v_{r_1,0}(u_1,t_2)}{\partial u_1}du_1$$

$$-\sum_{r_2=1}^{\infty}\int_{\eta_2}^{t_2}\zeta_2^{r_2}\frac{\partial v_{0,r_2}(t_1,u_2)}{\partial u_2}du_2 + g(t_1,t_2,0,0)]e^{-b}$$

$$= [\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\zeta_1^{r_1}\zeta_2^{r_2}\int_{\eta_1}^{t_1}(v'_{r_1,r_2}(u_1,t_2) - v'_{r_1,r_2}(u_1,\eta_2))du_1$$

$$+\sum_{r_1=1}^{\infty}\zeta_1^{r_1}(v_{r_1,0}(\eta_1,t_2) - v_{r_1,0}(t_1,t_2))$$

$$+\sum_{r_2=1}^{\infty}\zeta_2^{r_2}(v_{0,r_2}(t_1,\eta_2) - v_{0,r_2}(t_1,t_2)) + g(t_1,t_2,0,0)]e^{-b}$$

$$= [\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\zeta_1^{r_1}\zeta_2^{r_2}(v_{r_1,r_2}(t_1,t_2) - v_{r_1,r_2}(\eta_1,t_2))$$

$$-\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\zeta_1^{r_1}\zeta_2^{r_2}(v_{r_1,r_2}(t_1,\eta_2) - v_{r_1,r_2}(\eta_1,\eta_2))$$

$$+\sum_{r_1=1}^{\infty}\zeta_1^{r_1}v_{r_1,0}(\eta_1,t_2) + \sum_{r_2=1}^{\infty}\zeta_2^{r_2}v_{0,r_2}(t_1,\eta_2)$$

$$+ v_{0,0}(t_1,t_2)]e^{-b}$$

$$= [\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}v_{r_1,r_2}(\eta_1,\eta_2)\zeta_1^{r_1}\zeta_2^{r_2} + \sum_{r_1=1}^{\infty}v_{r_1,0}(\eta_1,t_2)\zeta_1^{r_1}$$

$$+\sum_{r_2=1}^{\infty}v_{0,r_2}(\eta_1,\eta_2)\zeta_2^{r_2} + v_{0,0}(t_1,t_2)]e^{-b}$$

$$= \Big[\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} v_{r_1,r_2}(\eta_1,\eta_2)\zeta_1^{r_1}\zeta_2^{r_2}\Big]e^{-b}$$

$$= \Big[\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} \zeta_1^{r_1}\zeta_2^{r_2}\sum_{j_1=0}^{r_1}\sum_{j_2=0}^{r_2} u_{j_1,j_2}(\eta_1,\eta_2)\frac{(n_1(t_1-\eta_1))^{r_1-j_1}}{(r_1-j_1)!}$$

$$\cdot\frac{(n_2(t_2-\eta_2))^{r_2-j_2}}{(r_2-j_2)!}\Big]e^{-b}$$

$$= \Big[\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} \zeta_1^{r_1}\zeta_2^{r_2} u_{r_1,r_2}(\eta_1,\eta_2)\Big]$$

$$\cdot\Big[\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} \zeta_2^{r_1}\frac{(n_1(t_1-\eta_1))^{r_1}}{r_1!}\zeta_2^{r_2}\frac{(n_2(t_2-\eta_2))^{r_2}}{r_2!}\Big]e^{-b}$$

$$= \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} u_{r_1,r_2}(\eta_1,\eta_2)\zeta_1^{r_1}\zeta_2^{r_2}. \qquad (2.3.13)$$

Furthermore as in the one sample case, only functions of (u_1,u_2,r_1,r_2) which are continuous in u_1 and u_2 for $\eta_1 < u_1 < t_1$ and $\eta_2 < u_2 < t_2$, $(r_1,r_2) \neq (0,0)$ can be unbiased estimators of differentiable (in η_1 and η_2) parametric functions that satisfy all the assumptions stated in the theorem. □

We consider next the case when $\eta_1=\eta_2=\eta$ but ζ_1 and ζ_2 are not necessarily equal. Again, we characterized estimable functions of the form $h(\eta,\zeta_1,\zeta_2)$. With this end, first the following theorem is proved. Let $z = \min(x_{(11)},x_{(21)})$ and for definiteness, let $t_1 < t_2$.

<u>Theorem 2.3.4</u> (Z,R_1,R_2) <u>is complete sufficient for</u> (η,ζ_1,ζ_2).

<u>Proof</u> First via Lemma 1.1.9 in Chapter One, we write the joint pdf of $X_{(11)},\ldots,X_{(1R_1)},X_{(21)},\ldots,X_{(2R_2)}$, R_1 and R_2 as

$$f(x_{(11)},\ldots,x_{(1r_1)},x_{(21)},\ldots,x_{(2r_2)},r_1,r_2)$$
$$= \prod_{i=1}^{2} \{(n_i\zeta_i)^{r_i}\exp(-n_i\zeta_i(t_i-\eta))$$
$$\cdot I_{[\eta<x_{(i1)}<\ldots<x_{(ir_i)}<t_i]},\} \; r_1>0,\; r_2>0; \tag{2.3.14}$$

$$f(x_{(21)},\ldots,x_{(2r_2)},0,r_2)$$
$$= (n_2\zeta_2)^{r_2}\exp(-\sum_{i=1}^{2} n_i\zeta_i(t_i-\eta))$$
$$\cdot I_{[\eta<x_{(21)}<\ldots<x_{(2r_2)}<t_2]},\; r_2>0; \tag{2.3.15}$$

$$f(x_{(11)},\ldots,x_{(1r_1)},r_1,0)$$
$$= (n_1\zeta_1)^{r_1}\exp(-\sum_{i=1}^{2} n_i\zeta_i(t_i-\eta))$$
$$\cdot I_{[\eta<x_{(11)}<\ldots<x_{(1r_1)}<t_1]},\; r_1>0; \tag{2.3.16}$$

$$f(0,0) = \exp(-\sum_{i=1}^{2} n_i\zeta_i(t_i-\eta)). \tag{2.3.17}$$

Note that in (2.3.15), and (2.3.16), one can write the indicators as $I_{[\eta<z]}$ since in the former case $z = \min(t_1,x_{(21)})$, while in the latter case $z = \min(x_{(11)},t_2)$. Thus, it is easy to see from (2.3.14) - (2.3.17) that (Z,R_1,R_2) is sufficient for (η,ζ_1,ζ_2).

To prove the completeness of Z,R_1 and R_2, we need first their joint pdf denoted here by $q(z,r_1,r_2)$. To this end, consider $\eta < z < \min(t_1,t_2) = t_1$. Also, we will write $u_1 = x_{(11)}$ and

$u_2 = x_{(21)}$. Then by independence of groups $P(Z > z,\ R_1=r_1, R_2=r_2)$

$$= P(U_1 > z,\ U_2 > z \mid R_1=r_1, R_2=r_2)P(R_1=r_1)P(R_2=r_2)$$
$$= P(U_1 > z \mid R_1=r_1)P(R_1=r_1)P(U_2 > z \mid R_2=r_2)P(R_2=r_2)$$
$$= [P(z<U_1<t_1 \mid R_1=r_1)+P(U_1=t_1 \mid R_1=r_1)]P(R_1=r_1)$$
$$\cdot[P(z<U_2<t_2 \mid R_2=r_2)+P(U_2=t_2 \mid R_2=r_2)]P(R_2=r_2) \qquad (2.3.18)$$

Hence, using (2.2.3) - (2.2.4) in (2.3.18) and the fact that marginally $R_i \sim$ Poisson $(n_i\zeta_i(t_i-\eta))$ $(i=1,2,\ldots)$, and simplifying, one gets for $\eta < z < t_1$.

$P(Z > z,\ R_1=r_1,\ R_2=r_2)$

$$= \frac{(n_1\zeta_1(t_1-z))^{r_1}(n_2\zeta_2(t_2-z))^{r_2}}{r_1!\ r_2!}\exp[-\sum_{i=1}^{2} n_1\zeta_1(t_1-\eta)]$$

$$\text{for } r_1 \geq 1,\ r_2 \geq 1; \qquad (2.3.19)$$

$$= \frac{(n_1\zeta_1\ (t_1-z)^{r_1})}{r_1!}\exp[-\sum_{i=1}^{2} n_i\zeta_i(t_i-\eta)] \quad r_1\geq 1,\ r_2=0; \qquad (2.3.20)$$

$$= \frac{(n_2\zeta_2(t_2-z))^{r_2}}{r_2!}\ \exp[-\sum_{i=i}^{2} n\ \zeta_i(t_i-\eta_i)] \quad r_1=0, r_2\geq 1; \qquad (2.3.21)$$

$$= \exp[-\sum_{i=1}^{2} (n_i\zeta_i(t_i-\eta)] \qquad r_1=0,\ r_2=0. \qquad (2.3.22)$$

Hence, by considering

$$P(Z<z \mid R_1=r_1, R_2=r_2)P(R_1=r_1, R_2=r_2)$$
$$= (1-P(Z>z \mid R_1=r_1, R_2=r_2))P(R_1=r_1, R_2=r_2)$$

and taking derivates in (2.3.19) through (2.3.22) with respect to z for $z\varepsilon(\eta,t_1)$ we obtain, that for $\eta<z<t_1$,

$q(z,r_1,r_2)$

$$= \left[\frac{(t_1-z)^{r_1-1}}{(r_1-1)!}\frac{(t_2-z)^{r_2}}{r_2!} + \frac{(t_1-z)^{r_1}}{r_1!}\frac{(t_2-z)^{r_2-1}}{(r_2-1)!}\right]$$

$$\cdot \prod_{i=1}^{2}(n_i\zeta_i)^{r_i}\exp[-\sum_{i=1}^{2} n_i\zeta_i(t_i-\eta)] \quad r_1\geq 1,\ r_2\geq 1 \quad (2.3.23)$$

$$= (n_2\zeta_2)^{r_2}\frac{(t_2-z)^{r_2-1}}{(r_2-1)!}\exp[\sum_{i=1}^{2} - n_i\zeta_i(t_i-\eta)] \quad r_1=0, r_2\geq 1 \quad (2.3.24)$$

$$= (n_1\zeta_1)^{r_1}\frac{(t_1-z)^{r_1-1}}{(r_1-1)!}\exp[\sum_{i=1}^{2} n_i\zeta_i(t_i-\eta)] \quad r_1\geq 1, r_2=0 \quad (2.3.25)$$

For $z = t_1$

$P(\min_{i=1,2} U_i=t_1,R_1=r_1,R_2=r_2)$

$$= P(Z=t_1|R_1=r_1,R_2=r_2)P(R_1=r_1)P(R_2=r_2)$$

$$= [P[U_1=t_1,U_2=t_1|R_1=r_1,R_2=r_2] + P[U_1=t_1,U_2>t_1|R_1=r_1,R_2=r_2]$$

$$+ P[U_1>t_1,U_2=t_1|R_1=r_1,R_2=r_2]]P(R_1=r_1,R_2=r_2)$$

$$= P(U_1=t_1,U_2>t_1|R_1=r_1,R_2=r_2)P(R_1=r_1,R_2=r_2)$$

$$= P(U_1=t_1|R_1=r_1)P(U_2>t_1|R_2=r_2)P(R_1=r_1)P(R_2=r_2)$$

Thus,

$P(\min_{i=1,2} U_i= t_1,R_1=r_1,R_2=r_2)$

$$= \exp[-n_1\zeta_1(t_1-\eta) - n_2\zeta_2(t_2-\eta)] \quad r_1=0, r_2=0 \quad (2.3.26)$$

$$= \frac{(t_2-t_1)^{r_2}}{r_2!} (n_2\zeta_2)^{r_2} \exp[-\sum_{i=1}^{2} n_i\zeta_i(t_i\eta)] \qquad r_1=0, r_2\geq 1. \quad (2.3.27)$$

From (2.2.23) - (2.3.27) it follows that if

$Eg(Z,R_1,R_2) = 0$ identically in $\eta\varepsilon(0,t_1]$, $\zeta_1>0, \zeta_2>0$ then

one must have

$$\Sigma_{r_1=1}^{\infty}\Sigma_{r_2=1}^{\infty}\int_{\eta}^{t_1} g(z,r_1,r_2) \pi_{i=1}^{2} (n_i\zeta_i)^{r_i}\left\{\frac{(t_1-z)^{r_1-1}}{(r_1-1)!}\frac{(t_2-z)^{r_2}}{r_2!}\right.$$

$$\left. + \frac{(t_1-z)^{r_1}}{r_1!}\frac{(t_2-z)^{r_2-1}}{(r_2-1)!}\right\}dz$$

$$+ \Sigma_{r_2=1}^{\infty}\int_{\eta}^{t_1} g(z,0,r_2)[(n_2\zeta_2)^{r_2}(t_2-z)^{r_2-1}/(r_2-1)!]dz$$

$$+ \Sigma_{r_2=1}^{\infty} g(t_1,0,r_2)(n_2\zeta_2)^{r_2}(t_2-t_1)^{r_2}/r_{2!}$$

$$+ \Sigma_{r_1=1}^{\infty}\int_{\eta}^{t_1} g(z,r_1,0)[(n_1\zeta_1)^{r_1}(t_1-z)^{r_1-1}/(r_1-1)!]dz$$

$$+ g(t_1,0,0) = 0. \qquad (2.3.28)$$

for all $\eta\varepsilon(0,t_1]$, $\zeta_1>0$ and $\zeta_2>0$. The left hand side being a bivariate power series in ζ_1 and ζ_2, equating the coefficient of $\zeta_1^{r_1}\zeta_2^{r_2}(r_1\geq 1,\ r_2\geq 1)$ on both sides, one gets

$$f(\eta)=\int_{\eta}^{t_1} g(z,r_1,r_2)\{r_2^{-1}(t_1-z)^{r_1-1}(t_2-z)^{r_2} +$$

$$+ r_1^{-1}(t_1-z)^{r_1}(t_2-z)^{r_2-1}\}dz = 0 \qquad (2.3.28a)$$

for all $\eta\varepsilon(0,t_1]$.

Fix $\eta < t$ choose $\eta_o\varepsilon(0,t]$ such that $\eta_o<\eta$. Then $f(\eta_o) = 0$ and

$$f(\eta_o)-f(\eta) = \int_{\eta_o}^{\eta} g(z,r_1,r_2)[r_2^{-1}(t_1-z)^{r_1-1}(t_2-z)^{r_2}$$

$$+ r_1^{-1}(t_1-z)^{r_1}(t_2-z)^{r_2-1}]dz = 0 \qquad (2.3.28b)$$

for all $\eta_o,\eta\varepsilon(0,t]$ such that $\eta_o<\eta$.

Write $g^1(z,r_1,r_2) = g(z,r_1,r_2)\{r_2^{-1}(t_1-z)^{r_1-1}(t_2-z)^{r_2}$

$$+ r_1^{-1}(t_1-z)^{r_1}(t_2-z)^{r_2-1}\} .$$

Then

$$\int_{\eta_o}^{\eta} g^1(z,r_1,r_2)dz = 0$$

for all $0 < \eta_o < \eta < t_1$

if and only if

$$\int_{\eta_o}^{\eta} g^{1+}(z,r_1 r_2)dz = \int_{\eta_o}^{\eta} g^{1-}(z,r_1,r_2)dz$$

if and only if

$$\int_B g^{1+}(z,r_1,r_2)dz = \int_B g^{1-}(z,r_1,r_2)dz \qquad \text{for all } B \varepsilon \mathfrak{B}$$

if and only if

$$g^{1+}(z,r_1,r_2) = g^{1-}(z,r_1,r_2) \quad \text{for all } \eta<z<t_1,r_1\geq 1,r_2\geq 1$$

if and only if $g^1(z,r_1,r_2) = 0$ which implies

$$g(z,r_1,r_2) = 0 \quad \text{for } \eta<z<t_1,\ r_1\geq 1,r_2\geq 1 \qquad (2.3.29)$$

Again equating coefficients on both sides of (2.3.28) for $\zeta_1^0\zeta_2^{r_2}$ $r_2\geq 1$, we obtain

$$\int_{\eta}^{t_1} g(z,0,r_2)n_2^{r_2} \frac{(t_2-z)^{r_2-1}}{(r_2-1)!} dz + g(t_1,0,r_2)n_2^{r_2} \frac{(t_2-t_1)^{r_2}}{r_2!} = 0$$

For fix η choose $\eta_o < \eta$, and proceed as before to obtain

$$\int_{\eta_o}^{\eta} g(z,0,r_2)n_2^{r_2} \frac{(t_2-z)^{r_2-1}}{(r_2-1)!} dz = 0 \quad \text{for all } 0<\eta_o<\eta<t_1$$

Hence, using the same argument

$$g(z,0,r_2) = 0 \quad \text{for all } \eta<z<t_1,\ r_2\geq 1 \tag{2.3.30}$$

Similarly one shows that

$$g(z,r_1,0) = 0 \qquad \eta<z<t_1,\ r_1\geq 1 \tag{2.3.31}$$

Using (2.3.29) - (2.3.31) and going back to (2.3.28) one gets

$$\sum_{r_2=0}^{\infty} g(t_1,0,r_2)(n_2\zeta_2)^{r_2} \frac{(t_2-t_1)^{r_2}}{r_2!} = 0$$

which by uniqueness of power series implies that

$$g(t_1,0,r_2) = 0 \text{ for all } r_2 \geq 0 \tag{2.3.32}$$

Hence, one must have $g(z,r_1,r_2) = 0$ a.e which completes the proof of the theorem. □

Arguing as in Section 2.2, it follows now that if $h(\eta,\zeta_1,\zeta_2)$ is estimable, then it must have a UMVUE based on Z, R_1 and R_2. Also, if there does not exist an unbiased estimator of $h(\eta,\zeta_1,\zeta_2)$ based on Z, R_1 and R_2, then $h(\eta,\zeta_1,\zeta_2)$ is not estimable.

The following theorem gives a necessary condition for estimability of $h(\eta,\zeta_1,\zeta_2)$.

<u>Theorem 2.3.5</u> $h(\eta,\zeta_1,\zeta_2)$ <u>is estimable only if it has the form</u> $\Sigma_{r_1=0}^{\infty}\Sigma_{r_2=0}^{\infty}u_{r_1,r_2}(\eta)\zeta_1^{r_1}\zeta_2^{r_2}$ <u>where</u> $u_{0,0}(\eta)$ <u>does not depend on</u> η and $u_{r_1,r_2}(t_1) = 0$ for $r_1 \geq 1$.

<u>Proof</u> Suppose $h(\eta,\zeta_1,\zeta_2)$ is estimable. Then, there exists a function $g(z,r_1,r_2)$ such that $Eg(Z,R_1,R_2) = h(\eta,\zeta_1,\zeta_2)$. Using (2.3.23)-(2.3.27) this implies that,

$h(\eta,\zeta_1,\zeta_2) =$

$$\sum_{r_1=1}^{\infty}\sum_{r_2=1}^{\infty}\int_{\eta}^{t_1} g(z,r_1,r_2)\Big[\frac{(t_1-z)^{r_1-1}}{(r_1-1)!}\,\frac{(t_2-z)^{r_2}}{r_2!}$$

$$+\frac{(t_1-z)^{r_1}}{r_1!}\,\frac{(t_2-z)^{r_2-1}}{(r_2-1)!}\Big]\prod_{i=1}^{2}(n_i\zeta_i)^{r_i}\exp\Big[-\sum_{i=1}^{2}n_i\zeta_i(t_i-\eta)\Big]dz$$

$$+\sum_{r_2=1}^{\infty}\int_{\eta}^{t_1} g(z,0,r_2)\frac{(t_2-z)^{r_2-1}}{(r_2-1)!}(n_2\zeta_2)^{r_2}\exp\Big[-\sum_{i=1}^{2}n_i\zeta_i(t_i-\eta)\Big]\,dz$$

$$+\sum_{r_1=1}^{\infty}\int_{\eta}^{t_1} g(z,r_1,0)\,\frac{(t_1-z)^{r_1-1}}{(r_1-1)!}(n_1\zeta_1)^{r_1}\exp\Big[-\sum_{i=1}^{2}n_i\zeta_i(t_i-\eta)\Big]dz$$

$$+\sum_{r_2=0}^{\infty} g(t_1,0,r_2)\,\frac{(t_2-t_1)^{r_2}}{r_2!}(n_2\zeta_2)^{r_2}\exp\Big[-\sum_{i=1}^{2}n_i\zeta_i(t_i-\eta)\Big] \qquad (2.3.33)$$

Note that for every fixed η, the right hand side is a bivariate power series in ζ_1 and ζ_2.

Hence $h(\eta,\zeta_1,\zeta_2)$ must be of the form

$$\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} u_{r_1,r_2}(\eta)\zeta_1^{r_1}\zeta_2^{r_2}.$$

For such a $h(\eta,\zeta_1,\zeta_2)$ with an unbiased estimator $g(Z,R_1,R_2)$, equating coefficients of $\zeta_1^0\zeta_2^0$ on both sides, one gets

$u_{0,0}(\eta) = g(t_1,0,0)$ for all $\eta\varepsilon(0,t_1]$

and $\sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} u_{r_1,r_2}(t_1)\zeta_1^{r_1}\zeta_2^{r_2}$

$$= \sum_{r_2=0}^{\infty} g(t_1 0, r_2) \frac{(t_2-t_1)^{r_2}}{r_2!} (n_2\zeta_2)^{r_2} \text{ which implies that}$$

$u_{r_1,r_2}(t_1) = 0$ for $r_1 > 1$. □

Remark 5 It follows from Theorem 2.3.5 that no nontrivial function $u(\eta)$ is estimable. Also, $k(\zeta_1,\zeta_2)$ is estimable only if it is a bivariate power series in ζ_1 and ζ_2. Thus, there does not exist any unbiased estimators of the scale parameters ζ_1^{-1} and ζ_2^{-1} .

The next theorem provides a necessary and sufficient condition for estimability of parametric functions of the form $\Sigma_{r_1=0}^{\infty}\Sigma_{r_2=0}^{\infty}u_{r_1,r_2}(\eta)\zeta_1^{r_1}\zeta_2^{r_2}$ which are differentiable in $\eta(<t_1)$.

Theorem 2.3.6 A parametric function $\Sigma_{r_1=0}^{\infty}\Sigma_{r_2=0}^{\infty}u_{r_1,r_2}(\eta)\zeta_1^{r_1}\zeta_2^{r_2}$ with $u_{r_1,r_2}(\eta)$ differentiable for all $\eta < t_1$ is estimable if and only if $u_{0,0}(\eta)$ does not depend on η and $u_{r_1,r_2}(t_1) = 0$ for $r_1 > 1$. Also, the class of estimators based on (Z,R_1,R_2) which are unbiased for differentiable parametric functions consists exactly of functions of (z,r_1,r_2) which are continous in z for $\eta < z < t_1$.

Proof Necessity has already been established the previous theorem. To prove sufficiency, suppose $g(z,r_1,r_2)$ is a continous function of $z\varepsilon(\eta,t_1)$.

Let

$$v_{r_1,r_2}(z) = \Sigma_{j_1=0}^{r_1}\Sigma_{j_2=0}^{r_2}u_{j_1,j_2}(z)\,\frac{(n_1(t_1-z))^{r_1-j_1}}{(r_1-j_1)!}\,\frac{(n_2(t_2-z))^{r_2-j_2}}{(r_2-j_2)!}$$

$$\eta < z < t_1. \qquad (2.3.34)$$

Then, from (2.3.23) - (2.3.24) and (2.3.34), it follows that

$$\Sigma_{r_1=0}^{\infty} \Sigma_{r_2=0}^{\infty} v_{r_1,r_2}(\eta) \zeta_1^{r_1} \zeta_2^{r_2}$$

$$= \Sigma_{r_1=1}^{\infty} \Sigma_{r_2=1}^{\infty} \zeta_1^{r_1} \zeta_2^{r_2} \int_{\eta}^{t_1} g(z,r_1,r_2) n_1^{r_1} n_2^{r_2}$$

$$\cdot \left\{ \frac{(t_1-z)^{r_1-1}}{(r_1-1)!} \frac{(t_2-z)^{r_2}}{r_2!} + \frac{(t_1-z)^{r_1}}{r_1!} \frac{(t_2-z)^{r_2-1}}{(r_2-1)!} \right\} dz$$

$$+ \Sigma_{r_2=1}^{\infty} \zeta_2^{r_2} \left[\int_{\eta}^{t_1} g(z,0,r_2) n_2^{r_2} \frac{(t_2-z)^{r_2-1}}{(r_2-1)!} dz \right.$$

$$\left. + g(t_1,0,r_2) n_2^{r_2} (t_2-t_1)^{r_2}/r_2! \right]$$

$$+ \Sigma_{r_1=1}^{\infty} \zeta_1^{r_1} \int_{\eta}^{t_1} g(z,r_1,0) n_1^{r_1} (t_1-z)^{r_1-1}/(r_1-1)! \, dz$$

$$+ g(t_1,0,0). \qquad (2.3.35)$$

Then using (2.3.35) and equating the coefficients of $\zeta_1^{r_1} \zeta_2^{r_2}$ on both sides, one gets

$$v_{r_1,r_2}(\eta) = \int_{\eta}^{t_1} g(z,r_1,r_2) n_1^{r_1} n_2^{r_2} \left\{ \frac{(t_1-z)^{r_1-1}}{(r_1-1)!} \frac{(t_2-z)^{r_2}}{r_2!} \right.$$

$$\left. + \frac{(t_1-z)^{r_1}}{r_1!} \frac{(t_2-z)^{r_2-1}}{(r_2-1)!} \right\} dz \text{ for } r_1 > 1,\ r_2 > 1; \qquad (2.3.36)$$

$$v_{0,r_2}(\eta) = \int_{\eta}^{t_1} g(z,0,r_2)(n_2^{r_2} (t_2-z)^{r_1-1}/(r_2-1)!)dz$$

$$+ g(t_1,0,r_2) n_2^{r_2} (t_2-t_1)^{r_2}/r_2! \qquad \text{for } r_2 > 1; \qquad (2.3.37)$$

$$v_{r_1,0}(\eta) = \int_\eta^{t_1} g(z,r_1,0)\left(\eta_1^{r_1} (t_1-z)^{r_1-1}/(r_1-1)!\right)dz, \qquad (2.3.38)$$

$$r_1 \geq 1;$$

$$v_{0,0}(\eta) = u_{0,0}(\eta) = g(t_1,0,0), \qquad (2.3.39)$$

for all $\eta < t_1$. Hence, from (2.3.36) - (2.3.38) differentiation with respect to η give for all $\eta < t_1$

$$g(\eta,r_1,r_2) = -\eta_1^{-r_1}\eta_2^{-r_2}\left\{\frac{(t_1-\eta)^{r_1-1}}{(r_1-1)!}\frac{(t_2-\eta)^{r_2}}{r_2!}\right.$$

$$\left.+\frac{(t_1-\eta)^{r_1}}{r_1!}\frac{(t_2-\eta)^{r_2-1}}{(r_2-1)!}\right\}v'_{r_1,r_2}(\eta) \text{ for all } r_1 \geq 1, r_2 \geq 1; \qquad (2.3.40)$$

$$g(\eta,0,r_2) = -\eta_2^{-r_2}(t_2-\eta)^{-(r_2-1)}(r_2-1)!\,v'_{0,r_2}(\eta),$$

$$\text{for all } r_2 \geq 1; \qquad (2.3.41)$$

$$g(\eta,r_1,0) = -\eta_1^{-r_1}(t_1-\eta)^{-(r_1-1)}(r_1-1)!\,v'_{r_1,0}(\eta) \text{ for all } r_1 \geq 1, \qquad (2.3.42)$$

Also, from (2.3.37) it follows that

$$v_{0,r_2}(t_1) = g(t_1,0,r_2)\eta_2^{r_2}(t_2-t_1)^{r_2}/r_2!, \quad r_2 \geq 1. \qquad (2.3.43)$$

Define $g(z,r_1,r_2)$ with η replaced by z in (2.3.40) - (2.3.42); then, using (2.3.34) - (2.3.35), (2.3.39) - (2.3.43) and the fact that $v_{r_1,r_2}(t_1) = u_{r_1,r_2}(t_1) = 0$ for $r_1 \geq 1$, one gets

$$Eg(Z,R_1R_2) = \Sigma_{r_1=0}^{\infty}\Sigma_{r_2=0}^{\infty}u_{r_1,r_2}(\eta)\zeta_1^{r_1}\zeta_2^{r_2}.$$

Hence, as before if $g_0(Z,R_1,R_2)$ is an unbiased estimator of $h(\eta,\zeta_1,\zeta_2)$ where $h(\eta,\zeta_1,\zeta_2)$ is differentiable in η, then we must have that $g_0(z,r_1,r_2)$ is continuous in z, because if it is not then by the first part of the theorem we can argue that $h(\eta,\zeta_1,\zeta_2)$ $= \Sigma_{r_1=0}^{\infty}\Sigma_{r_2=0}^{\infty} u_{r_1,r_2}(\eta)\zeta_1^{r_1}\zeta_2^{r_2}$ where $u_{0,0}(\eta)$ does not depend on η and $u_{r_1,r_2}(t_1) = 0$ for $r_1 > 1$. But, for such a function by the second part of the theorem, there exists an unbiased estimator which is continous in $z < t$. Hence, by completeness of (Z,R_1,R_2) the two estimators are equal. □

Thus in this case, the failure rate ζ_1 is not estimable since $\zeta_1 = \Sigma_{r_1=0}^{\infty}\Sigma_{r_2=0}^{\infty} u_{r_1,r_2}(\eta)\zeta_1^{r_1}\zeta_2^{r_2}$ where $u_{r_1,r_2}(\eta) = 1$ for $(r_1,r_2) = (1,0)$ and it is zero otherwise for all $0 < \eta \leq t_1$. Hence $u_{1,0}(t_1) = 1 \neq 0$ implies via the theorem that ζ_1 is not estimable. However, the other failure rate ζ_2 is estimable since

$$\zeta_2 = \sum_{r_1=0}^{\infty}\sum_{r_2=0}^{\infty} u_{r_1,r_2}(\eta)\zeta_1^{r_1}\zeta_2^{r_2} \text{ where } u_{r_1,r_2}(\eta) = 1 \text{ if}$$

$(r_1,r_2) = (0,1)$ and it is zero otherwise.

Hence, using (2.3.34) and (2.3.40) - (2.3.43) one gets that

$$v_{r_1,r_2}(z) = u_{0,1}(z)\,\frac{\left(n_1(t_1-z)\right)^{r_1}}{r_1!}\,\frac{\left(n_2(t_2-z)\right)^{r_1}}{(r_2-1)!}$$

$$= \frac{(n_1(t_1-z))^{r_1}}{r_1!}\frac{(n_2(t_2-z))^{r_2-1}}{(r_2-1)!} \qquad r_1>0,\ r_2>1$$

and

$g(z,r_1,r_2) =$

$$-n_1^{-r_1}n_2^{-r_2}\left\{\frac{(t_1-z)^{r_1-1}}{(r_1-1)!}\frac{(t_2-z)^{r_2}}{r_2!} + \frac{(t_1-z)^{r_1}}{r_1!}\frac{(t_2-z)^{r_2-1}}{(r_2-1)!}\right\}^{-1}$$

$$\cdot \frac{d}{dz}\left[\frac{(n_1(t_1-z))^{r_1}}{r_1!}\frac{(n_2(t_2-z))^{r_2-1}}{(r_2-1)!}\right]$$

$$= \frac{r_2(r_2-1)(t_1-z) + r_1r_2(t_2-z)}{n_2[r_2(t_1-z)(t_2-z) + r_1(t_2-z)^2]}$$

$$\text{for } r_1>0, r_2>1, \text{ and } \eta < z < t_1. \tag{2.3.44}$$

$$g(t_1,0,r_2) = \frac{r_2}{n_2(t_2-t_1)}, \qquad r_2 > 1 \tag{2.3.45}$$

<u>Remark 6</u> It is also clear from the above calculations that no power series of the form $\sum_{r_1=1}^{\infty} u_{r_1}(\eta)\zeta_1^{r_1}$ is estimable unless

$u_{r_1}(t_1) = 0$ for all $r_1 > 1$ and $u_0(\eta)$ does not depend on η. Thus there does not exist any nontrival estimable function involving η and ζ_1.

However, when ζ_2 is <u>known</u>, it can be easily seen from (2.3.14) - (2.3.17) that (Z,R_1) is sufficient for $(\eta,\zeta 1)$. Their joint pdf, obtained by summing r_2 out from (2.3.23) - (2.3.27), is given by

$$f(z,r_1) = [r_1+n_2\zeta_2(t_1-z)]\,\frac{(t_1-z)^{r_1-1}(n_1\zeta_1)^{r_1}}{r_1!}$$

$$\cdot\ \exp[-n_1\zeta_1(t_1-\eta)-n_2\zeta_2(z-\eta)] \quad \eta<z<t_1,\ r_1\geq 1; \tag{2.3.46}$$

$$= n_2\zeta_2\exp[-n_1\zeta_1(t_1-\eta)-n_2\zeta_2(z-\eta)] \quad \eta<z<t_1,\ r_1=0; \tag{2.3.47}$$

$$f(t_1,0) = \exp[-(n_1\zeta_1+n_2\zeta_2)(t_1-\eta)]; \tag{2.3.48}$$

The following corollary shows that this density is also complete. However, the proof is omitted since it is similar to the one given for Theorem 2.3.4.

Corallary 2.3.1 (Z,R_1) has a complete family of distributions. Once again our objective is to characterize estimable functions. To this end the following corollaries are obtained.

Corollary 2.3.2 When ζ_2 is known, a parametric function $h(\eta,\zeta_1)$ is estimable only if it has the form $\sum_{r_1=0}^{\infty} u_{r_1}(\eta)\zeta_1^{r_1}$ and $u_{r_1}(t_1) = 0$ for $r_1 > 1$.

Proof Suppose $g(Z,R_1)$ is unbiased for $h(\eta,\zeta_1)$, then from (2.3.46) - (2.3.48)

$$h(\eta,\zeta_1) = \sum_{r_1=0}^{\infty}\frac{(n_1\zeta_1)^{r_1}}{r_1!}\int_\eta^{t_1} g(z,r_1)[r_1+n_2\zeta_2(t_1-z)](t_1-z)^{r_1-1}$$

$$\cdot\ \exp[-n_1\zeta_1(t_1-\eta)-n_2\zeta_2(z-\eta)]dz$$

$$+ g(t_1,0)\exp[-(n_1\zeta_1+n_2\zeta_2)(t_1-\eta)]. \tag{2.3.49}$$

Since the right hand side is a power series in ζ_1, it follows that $h(\eta,\zeta_1)$ must be of the form $\sum_{r_1=0}^{\infty} u_{r_1}(\eta)\zeta_1^{r_1}$ and

$h(t_1,\zeta_1) = \sum_{r_1=0}^{\infty} u_{r_1}(t_1)\zeta_2^{r_1} = g(t_1,0)$ from which the result follows. Note that in this case $u_0(\eta)$ can depend on η.

Also, ζ_1^{-1} is not estimable since it can not be represented as a power series. □

Corollary 2.3.3 When ζ_2 is known, a parametric function $\sum_{r_1=0}^{\infty} u_{r_1}(\eta)\zeta_1^{r_1}$ which is differentiable in η for $\eta < t_1$ admits an unbiased estimator if and only if $u_{r_1}(t_1) = 0$ for $r_1 > 1$. Furthermore only continous functions of z can be unbiased for parametric functions that satisfy these assumptions.

Proof Necessity has already been proven on the previous corollary.

Hence, assume now that $g(z,r_1)$ is continous for $\eta < z < t_1$ and $u_{r_1}(t_1) = 0$. Then using (2.3.49), one gets

$$\sum_{r_1=0}^{\infty} \zeta_1^{r_1} \sum_{j_1=0}^{r_1} u_{j_1}(\eta) \frac{(n_1(t_1-\eta))^{r_1-j_1}}{(r_1-j_1)!} \exp[-n_2\zeta_2\eta]$$

$$= \sum_{r_1=0}^{\infty} \frac{(n_1\zeta_1)^{r_1}}{r_1!} \int_{\eta}^{t_1} g(z,r_1)[r_1+n_2\zeta_2(t_1-z)](t_1-z)^{r_1-1}$$

$$\times \exp[-n_2\zeta_2 z]dz + g(t_1,0)\exp(-n_2\zeta_2 t_1) \tag{2.3.50}$$

for all $\eta\varepsilon(0,t_1]$, $\zeta_1 > 0$.

Equating coefficients on both sides of (2.3.50) and for $\eta < t$ we obtain

$$v_{r_1}(\eta) = \sum_{j_1=0}^{\infty} \exp[-n_2\zeta_2\eta]u_{j_1}(\eta)\frac{(n_1(t_1-\eta))^{r_1-j_1}}{(r_1-j_1)!}$$

$$= \frac{n_1^{r_1}}{r_1!}\int_\eta^{t_1} g(z,r_1)[r_1+n_2\zeta_2(t_1-z)](t_1-z)^{r_1-1}\exp[-n_2\zeta_2 z]dz$$

for $r_1 \geq 1$; (2.3.51)

$$v_0(\eta) = \exp[-n_2\zeta_2\eta]u_0(\eta) = \int_\eta^{t_1} g(z,0)n_2\zeta_2\exp[-n_2\zeta_2 z]dz$$

$$+ g(t_1,0)\exp[-n_2\zeta_2 t_1] \tag{2.3.52}$$

Differenting both sides of (2.3.51) - (2.3.52) with respect to $\eta < t_1$, and putting z for η we obtain

$$\frac{d}{dz} v_{r_1}(z) = -\frac{n_1^{r_1}}{r_1!} g(z,r_1)[r_1+n_1\zeta_2(t_1-z)](t_1-z)^{r_1-1}\exp[-n_2\zeta_2 z] \tag{2.3.53}$$

$$\frac{d}{dz} v_0(z) = -g(z,0)n_2\zeta_2\exp[-n_2\zeta_2 z] \tag{2.3.54}$$

Also from (2.3.51) and (2.3.52),

$$v_0(t_1) = \exp[-n_2\zeta_2 t_1]u_0(t_1) = \exp[-n_2\zeta_2 t_1]g(t_1,0) \tag{2.3.55}$$

Using (2.3.53) - (2.3.54) and the fact that

$v_{r_1}(t_1) = u_{r_1}(t_1)e^{-n_2\zeta_2 t_1} = 0$, for $r_1 \geq 1$, one gets

$$Eg(Z,R_1) = \sum_{r_1=0}^{\infty} u_{r_1}(\eta)\zeta_1^{r_1}.$$

Using familiar arguments it follows that the class of unbiased estimators of differentiable (in η) parametric functions

consists exactly of functions of z and r_1, which are continous in z for $\eta < z < t_1$ and $r_1 \geq 1$. □

It follows from the theorem that ζ_1 is still not estimable. However $\eta = \sum_{r_1=0}^{\infty} u_{r_1}(\eta)\zeta_1^{r_1}$, where $u_r(\eta) = 1$ if $r_1 = 0$ and is zero otherwise, is estimable and using (2.3.51) - (2.3.55) one gets that the UMVUE is given by

$$g(Z,R_1) = \left(Z - \left(\frac{R_1}{(t_1-Z)} + n_2\zeta_2\right)^{-1}\right) I_{(\eta<Z<t_1, R_1>0)}$$

$$+ t_1 I_{(Z = t_1, R_1 = 0)}$$

When ζ_1 is <u>known</u> but η and ζ_2 are unknown, it may be easily seen from (2.3.14) - (2.3.17) that (Z,R_2) is sufficient for (η,ζ_2). Using (2.3.23) - (2.3.27) their joint pdf is given by

$$f(z,r_2) = [r_2 + n_1\zeta_1(t_2-z)] \frac{(n_2\zeta_2(t_2-z))^{r_2-1}}{r_2!}$$

$$\times \exp[-n_1\zeta_1(z-\eta) - n_2\zeta_2(t_2-\eta)] \qquad \eta<z<t_1,\ r_2\geq 1; \quad (2.3.56)$$

$$= n_1\zeta_1 \exp[-n_1\zeta_1(z-\eta) - n_2\zeta_2(t_2-\eta)] \qquad \eta<z<t_1, r_2=0. \quad (2.3.57)$$

$$P(Z = t_1, R_2=0) = \exp[-n_1\zeta_1(t_1-\eta) - n_2\zeta_2(t_2-\eta)] \quad (2.3.58)$$

$$P(Z=t_1, R_2=r_2) = \frac{(n_2\zeta_2(t_2-t_1))^{r_2}}{r_2!}$$

$$\times \exp[-n_1\zeta_1(t_1-\eta) - n_2\zeta_2(t_2-\eta)] \qquad r_2 \geq 1. \quad (2.3.59)$$

<u>Corollary 2.3.4</u> When ζ_1 is known, (Z,R_1) has a complete family of distributions.

The proof is omitted because of the similarity to the one given for Theorem 2.3.4.

The characterization of estimable functions which are differentiable in this situation is given by the next corollary.

<u>Corollary 2.3.5</u> When ζ_1 is known, a parameteric function $h(\eta,\zeta_2)$ which is differentiable in η admits an unbiased estimator based on functions of (Z,R_2) if and only if it has the form $\sum_{r_2=0}^{\infty} u_{r_2}(\eta)\zeta_2^{r_2}$.

Also, the class of unbiased estimators of differentiable parametric functions consists exactly of functions of z and r_2 that are continuous in z for $\eta<z<t_1$, $r_2 \geq 1$.

<u>Proof</u> Suppose $g(Z,R_2)$ is unbiased for $h(\eta,\zeta_2)$. Then, from (2.3.56) - (2.3.59)

$$h(\eta,\zeta_2) = \sum_{r_2=0}^{\infty} \frac{\zeta_2^{r_2}}{r_2!} \int_\eta^{t_1} g(z,r_2)[r_2+n_1\zeta_1(t_1-z)](t_2-z)^{r_2-1}$$

$$\times \exp[-n_1\zeta_1(z-\eta) - n_2\zeta_2(t_2-\eta)]dz$$

$$+ \sum_{r_2=0}^{\infty} g(t_1,r_2) \frac{(n_2(t_2-t_1))^{r_2}}{r_2!} \zeta_2^{r_2}$$

$$\times \exp[-n_1\zeta_1(t_1-\eta) - n_2\zeta_2(t_2-\eta)] \qquad (2.3.60)$$

for all $0 < \eta < t_1$, $\zeta_2 > 0$.

Again, the right hand side of (2.3.60) is a power series in ζ_2 therefore, $h(\eta,\zeta_2)$ must be of the form $\sum_{r_2=0}^{\infty} u_{r_2}(\eta)\zeta_2^{r_2}$.

Assume now that $h(\eta,\zeta_2) = \sum_{r_2=0}^{\infty} u_{r_2}(\eta)\zeta_2^{r_2}$ and $g(z,r_2)$ is continuous for $\eta < z < t_1$, $r_2 > 1$.

Then for $\eta\varepsilon[0,t_1]$ and using (2.3.60)

$$\sum_{r_2=0}^{\infty} \zeta_2^{r_2} \sum_{j_2=0}^{r_2} u_{j_2}(\eta) \frac{(n_2(t_2-\eta))^{r_2-j_2}}{(r_2-j_2)!} \exp[-n_1\zeta_1\eta]$$

$$= \sum_{r_2=0}^{\infty} \frac{(n_2\zeta_2)^{r_2}}{r_2!} \{\int_\eta^{t_1} g(z,r_2)[r_2+n_1\zeta_1(t_2-z)](t_2-z)^{r_2-1}$$

$$\times \exp[-n_1\zeta_1 z]dz + g(t_1,r_2)(t_2-t_1)\exp[-n_1\zeta_1 t_1]\} \qquad (2.3.61)$$

Equating coefficients on both sides of (2.3.61), we obtain

$$v_{r_2}(\eta) = \sum_{j_2=0}^{r_2} u_{j_2}(\eta) \frac{(n_2(t_2-\eta))^{r_2-j_2}}{(r_2-j_2)!} e^{-n_1\zeta_1\eta} \qquad (2.3.62)$$

$$= \int_\eta^{t_1} \frac{n_2^{r_2}}{r_2!} g(z,r_2)[r_2+n_1\zeta_1(t_2-z)](t_2-z)^{r_2-1} \exp[-n_1\zeta_1 z]dz$$

$$+ g(t_1,r_2) \frac{(n_2(t_2-t_1))^{r_2}}{r_2!} \exp[-n_1\zeta_1 t_1],\ r_2>0 \qquad (2.3.63)$$

for $\eta\varepsilon(0,t_1]$.

Differentiating both sides of (2.3.63) with respect to $\eta < t_1$ and putting z for η, one gets

$$\frac{d}{dz} v_{r_2}(z) = \frac{n_2^{r_2}}{r_2!} g(z,r_2)[r_2+n_1\zeta_1(t_2-z)](t_2-z)^{r_2-1} \exp[-n_1\zeta_1 z]$$

$$\text{for } r_2 > 0. \qquad (2.3.64)$$

Also from (2.3.60) with $h(\eta,\zeta_2) = \sum_{r_2=0}^{\infty} u_{r_2}(\eta)\zeta_2^{r_2}$

$$v_{r_2}(t_1) = g(t_1,r_2) \frac{(n_2(t_2-t_1))^{r_2}}{r_2!} \exp[-n_1\zeta_1 t_1],\ r_2>0. \qquad (2.3.65)$$

Hence, using (2.3.64) - (2.3.65), one gets

$$Eg(Z,R_2) = \sum_{r_2=0}^{\infty} u_{r_2}(\eta)\zeta_2^{r_2}.$$

Again, familiar arguments show that only functions that are continous in z qualify as estimators of differentiable parametric functions. Also note that if a function $h(\eta_1,\zeta_2)$ is estimable but not necessarily differentiable, then we must have that it is a power series in ζ_2. □

Hence, it follows using (2.3.62) and (2.3.64) - (2.3.65) that η is estimable with UMVUE given by the function

$$g(Z,R_2) = Z - \left(\frac{R_2}{(t_2-Z)} + n_1\zeta_1\right)I_{(\eta < Z < t_1,\ R_2 > 0)}$$

Also ζ_2 is estimable, again, using (2.3.62) and (2.3.64)-(2.3.65); its UMVUE is given by

$$g(Z,R_2) = \frac{\dfrac{n_2R_2(R_2-1)}{[n_2(t_2-Z)]^2} + n_1\zeta_1\dfrac{R_2}{n_2(t_2-Z)}}{\dfrac{R_2}{(t_2-Z)} + n_1\zeta_1} I_{(\eta<Z<t_1,R_2>1)}$$

$$+ \frac{R_2}{n_2(t_2-t_1)} I_{(Z = t_1,\ R_2 > 0)}$$

This apparent anamoly in the behavior of power series in ζ_1 and ζ_2 is due to the fact that the censoring times t_1 and t_2 are distinct and $t_1<t_2$. The situation reverses when $t_2 < t_1$.

If both ζ_1 and ζ_2 are known, then only Z is sufficient for η, its pdf may be obtained from (2.3.46)-(2.3.48) or (2.3.56)-(2.3.59) and is given by

$$f(z) = (n_1\zeta_1+n_2\zeta_2)\exp[-(n_1\zeta_1+n_2\zeta_2)(z-\eta)] \qquad \eta<z<t_1; \tag{2.3.66}$$

$$P(Z = t_1) = \exp[-(n_1\zeta_1+n_2\zeta_2)(t_1-\eta)] \tag{2.3.67}$$

This case is very similar to the one sample situation when ζ is known. Hence, any differentiable function $\mu(\eta)$ admits an unbiased estimator if and only if $\mu(t_1) = g(t_1)$. Also only continuous functions of z can be estimators for differentiable parametric. In such case the UMVUE is given by

$$g(Z) = [u(Z) - u'(Z)(n_1\zeta_1+n_2\zeta_2)^{-1}]I_{[\eta<Z<t]} \tag{2.3.68}$$

$$g(t_1) = \mu(t_1) \tag{2.3.69}$$

Hence, the UMVUE for η is given by

$$Z - (n_1\zeta_1+n_2\zeta_2)^{-1}I_{[\eta<Z<t]}$$

If a function $u(\eta)$ is estimable but not differentiable (in $\eta < t$) then $u(t_1) = g(t_1)$

Remark 7 If η is known, then from (2.3.14)-(2.3.17) (R_1,R_2) is sufficient for (ζ_1,ζ_2), its pdf is given by

$$P(R_1=r_1,\ R_2=r_2) = \frac{(n_1\zeta_1(t_1-\eta))^{r_1}}{r_1!}\ \frac{(n_2\zeta_2(t_2-\eta))^{r_2}}{r_2!}$$

$$\times \exp[-n_1\zeta_1(t_1-\eta)-n_2\zeta_2(t_2-\eta)] \tag{2.3.70}$$

which is the joint density of two independent Poisson variables.

Hence, using arguments similar to those used in all other cases, it follows that a parametric function is estimable if and only if it is a bivariate power series in ζ_1 and ζ_2. In particular, both ζ_1 and ζ_2 are estimable. Indeed the UMVUEs of ζ_1 and ζ_2 are given respectively by $\tilde{\tilde{\zeta}}_1$ and $\tilde{\tilde{\zeta}}_2$ with
$\tilde{\tilde{\zeta}}_i = R_i/n_i(t_i-\eta)$ $(i=1,2)$.

In general the UMVUE of any function of the form

$\sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} a_{r_1,r_2} \zeta_1^{r_1} \zeta_2^{r_2}$ is given by

$$g(R_1,R_2) = \frac{R_1!}{\left(n_1(t_1-\eta)\right)^{R_1}} \frac{R_2!}{\left(n_2(t_2-\eta)\right)^{R_2}}$$

$$\times \sum_{j_1=0}^{R_1} \sum_{j_2=0}^{R_2} a_{j_1,j_2} \frac{\left(n_1(t_1-\eta)\right)^{R_1-j_1}}{(R_1-j_1)!} \frac{\left(n_2(t_2-\eta)\right)^{R_2-j_2}}{(R_2-j_2)!} ; \qquad (2.3.71)$$

Next we consider the case when η_1 and η_2 are not necessarily equal but $\zeta_1 = \zeta_2 = \zeta$. Let $R = R_1 + R_2$. It is easy to see from (2.3.14)-(2.3.17) that in this case $(X_{(11)}, X_{(21)}, R)$ is sufficient for (η_1,η_2,ζ). Also, $R \sim$Poisson $(c\zeta)$ where $c = c(\eta)$ $= \sum_{i=1}^{2} n_i(t_i-\eta_i)$.

To obtain the joint density of $(X_{(11)}, X_{(21)}, R)$, write $W = R_2$, $U_1 = X_{(11)}$, $U_2 = X_{(21)}$. Then $R_1 = R-W$. Using (2.3.1), one gets $g(u_1,u_2,r,w) = f(u_1,u_2,r-w,w)$

$$= (r-w)\,\frac{(t_1-u_1)^{r-w-1}}{(r-w)!}\,\frac{w(t_2-u_2)^{w-1}}{w!}\,(n_1\zeta_1)^{r-w}(n_2\zeta_2)^{w}\exp[-c\zeta]$$

$$\text{for } w>0,\ r-w>0,\ \eta_1<u_1<t_1,\ \eta_1<u_2<t_2; \qquad (2.3.72)$$

$$= r\,\frac{(t_1-u_1)^{r-1}}{r!}\,(n_1\zeta)^r\,e^{-c\zeta} \qquad r\geq 1,\ w=0,\ u_2=t_2; \qquad (2.3.73)$$

$$= r\,\frac{(t_2-u_2)^{r-1}}{r!}\,(n_2\zeta)^r\,e^{-c\zeta} \qquad w=r\geq 1,\ u_1=t_1; \qquad (2.3.74)$$

$$= e^{-c\zeta} \qquad r=0,\ w=0; \qquad (2.3.75)$$

For $r \geq 2$

$$q(u_1,u_2,r) = \sum_{w=1}^{r-1} f(u_1,u_2,r-w,w)$$

$$= \sum_{w=1}^{r-1} \frac{(n_1\zeta)^{r-w}(n_2\zeta)^{r}(t_1-u_1)^{r-w-1}(t_2-u_2)^{w-1}}{(r-w-1)!\,(w-1)!}\,e^{-c\zeta}$$

$$= e^{-c\zeta}(n_1\zeta)(n_2\zeta)\sum_{w=1}^{r-1} \frac{\big(n_1\zeta(t_1-u_1)\big)^{(r-2-(w-1))}\cdot\big(n_2\zeta(t_2-u_2)\big)^{w-1}}{\big(r-2-(w-1)\big)!\,(w-1)!}$$

$$= e^{-c\zeta}\,\frac{n_1n_2\zeta^2}{(r-2)!}\sum_{w=1}^{r-1}\binom{r-2}{w-1}\big(n_1\zeta(t_1-u_1)\big)^{(r-2-(w-1))}\big(n_2\zeta(t_2-u_2)\big)^{w-1}$$

$$= e^{-c\zeta}\,\frac{n_1n_2\zeta^2}{(r-2)!}\big(n_1\zeta(t_1-u_1) + n_2\zeta(t_2-u_2)\big)^{r-2}. \qquad (2.3.76)$$

Combining (2.3.71) - (2.3.76), one gets

$$q(u_1,u_2,r) = e^{-c\zeta}\frac{n_1n_2\zeta^r}{(r-2)!}\big(n_1(t_1-u_1) + n_2(t_2-u_2)\big)^{r-2}$$

$$r\geq 2,\ \eta_i<u_i<t_i,\ i=1,2 \qquad (2.3.77)$$

$$= r\,\frac{(t_1-u_1)^{r-1}}{r!}\,(n_1\zeta)^r\,e^{-c\zeta} \qquad r\geq 1,\ u_2=t_2,\ \eta_1<u_1<t_1; \qquad (2.3.78)$$

$$= r\,\frac{(t_2-u_2)^{r-1}}{r!}\,(\eta_2\zeta)^r\,e^{-c\zeta} \qquad r\geq 1,\ u_1=t_1,\ \eta_2<u_2<t_2; \quad (2.3.79)$$

$$= e^{-c\zeta} \qquad r=0 \iff u_1=t_1,\ u_2=t_2. \quad (2.3.80)$$

<u>Theorem 2.3.7</u> (U_1,U_2,R) <u>is complete for</u> (η_1,η_2,ζ).
Again this proof is omitted because of the similarity to the one given for Theorem 2.3.4.

The next theorem provides a necessary condition for estimability of $h(\eta_1,\eta_2,\zeta)$.

<u>Theorem 2.3.8</u> $h(\eta_1,\eta_2,\zeta)$ <u>is estimable only if it is of the form</u> $\sum_{r=0}^{\infty} u_r(\eta_1,\eta_2)\zeta^r$, <u>where</u> $u_0(\eta_1,\eta_2)$ <u>does not depend on</u> η_1 <u>and</u> η_2 and $u_r(t_1,t_2) = 0$ for every $r \geq 1$.

<u>Proof</u> It follows from (2.3.77)-(2.3.80) that $h(\eta_1,\eta_2,\zeta)$ is estimable only if for each fixed η_1 and η_2, it is a power series in ζ. Hence, an estimable $h(\eta_1,\eta_2,\zeta)$ must be of the form $\sum_{r=0}^{\infty} u_r(\eta_1,\eta_2)\zeta^r$. From (2.3.77)-(2.3.80) it follows also that if $g(U_1,U_2,R)$ is an unbiased estimator of $\sum_{r=0}^{\infty} u_r(\eta_1,\eta_2)\zeta^r$, then equating the coefficients of ζ^0 from two identical power series in $\zeta(>0)$, one gets $g(t_1,t_2,0) = u_0(\eta_1,\eta_2)$ for all $\eta_1\varepsilon(0,t_1]$ and $\eta_2\varepsilon(0,t_2]$ so that $u_0(\eta_1,\eta_2)$ does not depend on η_1 and η_2. Also for $(\eta_1,\eta_2) = (t_1,t_2)$ one gets $\sum_{r=0}^{\infty} u_r(t_1,t_2) = g(t_1,t_2,0)$ from which the conclusion follows. □

A necessary and sufficient condition for estimability of $\sum_{r=0}^{\infty} u_r(\eta_1,\eta_2)\zeta^r$ where each $u_r(\eta_1,\eta_2)$ is differentiable in both η_1 and η_2 is given below.

Theorem 2.3.9 <u>Suppose</u> $u_r(\eta_1,\eta_2)$ <u>is differentiable for every</u> $r \geq 1$. <u>Then</u> $\sum_{r=0}^{\infty} u_r(\eta_1,\eta_2)\zeta^r$ <u>is estimable if and only if</u> $u_0(\eta_1,\eta_2)$ <u>does not depend on</u> η_1 <u>and</u> η_2 <u>and</u> $u_r(t_1,t_2) = 0$ <u>for every</u> $r \geq 1$. <u>In such case only functions of</u> u_1,u_2 <u>and</u> r <u>which are continuous in</u> $\eta_1 < u_1 < t_1$ and $\eta_2 < u_2 < t_2$ for $r \geq 1$ <u>can be unbiased estimators of parametric functions of the form</u> $\sum_{r=0}^{\infty} u_r(\eta_1,\eta_2)\zeta^r$ <u>which are differentiable in</u> η_1 <u>and</u> η_2.

<u>Proof</u> Again, necessity has already been established. To show sufficiency, suppose $g(u_1,u_2,r)$ is continuous in $u_1 \varepsilon (\eta_1,t_1)$ and $u_2 \varepsilon (\eta_2,t_2)$ for $r \geq 1$. Then, it follows from (2.3.77)-(2.3.80) that

$$[\sum_{r=0}^{\infty} u_r(\eta_1,\eta_2)\zeta^r][\sum_{r=0}^{\infty} \frac{\{\Sigma_{i=1}^2 \eta_i(t_i-\eta_i)\}^r}{r!}\zeta^r]$$

$$= g(t_1,t_2 0) + \sum_{r=1}^{\infty} \zeta^r((r-1)!)^{-1}\{\int_{\eta_1}^{t_1} \eta_1^r g(u_1,t_2,r)(t_1-u_1)^{r-1}du_1$$

$$+ \int_{\eta_2}^{t_2} \eta_2^r g(t_1,u_2,r)(t_2-u_2)^{r-1}du_2\}$$

$$+ \sum_{r=2}^{\infty} \{\zeta^r/(r-2)!\}\eta_1\eta_2\int_{\eta_2}^{t_2}\int_{\eta_1}^{t_1} g(u_1,u_2,r)(\sum_{i=1}^{\infty} \eta_i(t_i-u_i))^{r-2}du_1du_2 \tag{2.3.81}$$

for all $\eta_1 < t_1$, $\eta_2 < t_2$ and $\zeta > 0$.

Let

$$v_r(\eta_1,\eta_2) = \sum_{j=0}^{r} u_j(\eta_1,\eta_2)\frac{[\Sigma_{i=1}^2 \eta_i(t_i-\eta_i)]^{r-j}}{(r-j)!} \quad r=0,1,\ldots \tag{2.3.82}$$

Then equating the coefficients of ζ^r on both sides of (2.3.81), one gets

$$v_0(\eta_1,\eta_2) = g(t_1,t_2,0); \tag{2.3.83}$$

$$v_1(\eta_1,\eta_2) = n_1\int_{\eta_1}^{t_1} g(u_1,t_2,1)du_1+n_2\int_{\eta_2}^{t_2} g(t_1,u_2,1)du_2; \tag{2.3.84}$$

$$v_r(\eta_1,\eta_2)=((r-2)!)^{-1}n_1n_2\int_{\eta_2}^{t_2}\int_{\eta_1}^{t_1} g(u_1,u_2,r)(\Sigma_{r=1}^2 n_i(t_i-u_i))^{r-2}du_1du_2$$

$$+ ((r-1)!)^{-1}[\int_{\eta_1}^{t_1} n_1^r g(u_1,t_2,r)(t_1-u_1)^{r-1}du_1$$

$$+ \int_{\eta_2}^{t_2} n_2^r g(t_1,u_2,r)(t_2-u_2)^{r-1}du_2], \tag{2.3.85}$$

for all $\eta_1 < t_2$, $\eta_2 < t_2$ and $r > 2$. It follows from (2.3.84) that

$$v_1(\eta_1,t_2) = n_1 \int_{\eta_1}^{t_1} g(u_1,t_2,1)du_1,$$

$$v_1(t_1,\eta_2) = n_2\int_{\eta_2}^{t_2} g(t_1,u_2,1)du_2, g(u_1,t_2,1)=-n_1^{-1}\partial v_1(u_1,t_2)/\partial u_1,$$

and $g(t_1,u_2,1) = -n_2^{-1}\partial v_1(t_1,u_2)/\partial u_2$ for all $u_1<t_1$, $u_2<t_2$. Again, from (2.3.85), one gets,

$n_1^r g(u_1,t_2,r)(t_1-u_1)^{r-1}/(r-1)! = \partial v_1(u_1,t_2)/\partial u_1$ for $u_1<t_1$ and

$n_2^r g(t_1,u_2,r)(t_2-u_2)^{r-1}/(r-1)!=\partial v_r(t_1,u_2)/\partial u_2$ for $u_2<t_2$.

Further, from (2.3.85)

$((r-2)!)^{-1}n_1n_2g(u_1,u_2,r)(\Sigma_{i=1}^2 n_i(t_i-u_i))^{r-2}=\partial^2 v_r(u_1,u_2)/\partial u_1\partial u_2$

for $u_1<t_1$, $u_2<t_2$. Moreover, using the fact that

$$v_r(t_1,t_2) = u_r(t_1,t_2) = 0 \text{ for } r > 1. \tag{2.3.86}$$

It follows that the estimator $g(U_1,U_2,R)$ with

$$g(t_1,t_2,0) = v_0(t_1,t_2) = u_0(t_1,t_2)$$

$$g(u_1,t_2,r) = -n_1^{-r}(t_1-u_1)^{-(r-1)}(r-1)!\left(\partial v_r(u_1,t_2)/\partial u_1\right), \quad r > 1,$$

$$g(t_1,u_2,r) = -n_2^{-r}(t_2-u_2)^{-(r-1)}(r-1)!\left(\partial v_r(t_1,u_2)/\partial u_2\right), \quad r > 1,$$

and

$$g(u_1,u_2,r) =$$

$$(n_1n_2)^{-1}\left(\Sigma_{i=1}^2 n_i(t_i-u_i)\right)^{-(r-2)}(r-2)!\left(\partial^2 v_1(u_1,u_2)/\partial u_1\partial u_2\right)$$

for $r > 2$ has expectation $\sum_{r=0}^{\infty} u_r(\eta_1,\eta_2)\zeta^r$.

Also, using what has been proven so far in the theorem and the completeness of U_1, U_2, and R, it follows that any function $g_0(u_1,u_2,r)$ which is unbiased for $h(\eta_1,\eta_2,\zeta)$ where $h(\eta_1,\eta_2,\zeta)$ is differentiable in η_1 ($<t_1$) and η_2 ($<t_2$) must be continuous in $u_1 \varepsilon (\eta_1,t_1)$ and $u_2 \varepsilon (\eta_2,t_2)$. □

This completes the proof of the theorem.

Thus, in this case, the failure rate ζ nor any function $u(\eta_1,\eta_2)$ depending only on η_1, η_2 or both admits an unbiased estimator based on any function of U_1, U_2, and R.

Remark 8 If, however, one of the parameters, say η_2 is known, it is easy to see from (2.3.14)-(2.3.17) that U_1 and R are sufficient for (η_1,ζ).

From (2.3.77)-(2.3.80) we obtain their joint pdf which is given by

$$f(u_1,r) = \exp(-c\zeta)n_1\left[n_1(t_1-u_1) + n_2(t_2-\eta_2)\right]^{r-1}\zeta^r/(r-1)!$$

$$\text{for } r > 1,\ \eta_1 < u_1 < t_1; \tag{2.3.87}$$

$$f(t_1,r) = \exp(-c\zeta)\,\left(n_2\zeta(t_2-\eta_2)\right)^r/r! \qquad r > 0. \qquad (2.3.88)$$

Here $c = \sum_{i=1}^{2} n_i(t_i-\eta_i)$.

It can also be shown that this density is complete.

It is easy to see from (2.3.87)-(2.3.88) that if a parametric function $h(\eta_1,\zeta)$ is estimable it must be of the form $\sum_{r=0}^{\infty} u_r(\eta_1)\zeta^r$ where $u_0(\eta_1) = g(t_1,0)$ does not depend on η_1.

Our next result shows that this is also a sufficient condition if $h(\eta_1,\zeta)$ is differentiable in $\eta_1 < t_1$.

Corollary 2.3.6 A parametric function $h(\eta_1,\zeta)$ differentiable in η_1 admits an unbiased estimator based on functions of (U_1,R) if and only if it is of the form $\sum_{r=0}^{\infty} u_r(\eta_1)\zeta^r$ where $u_0(\eta_1)$ does not depend on η_1. In such case, the only possible estimators of differentiable parametric functions are those that are continuous in u_1 for $\eta_1 < u_1 < t_1$, $r > 1$.

Proof Assume $h(\eta_1,\zeta) = \sum_{r=0}^{\infty} u_r(\eta_1)\zeta^r$ where $u_0(\eta_1)$ does not depend on η_1. Let $g(u_1,r)$ be continuous in u_1 for $u_1\varepsilon(0,t_1)$, $r > 1$. Then

$$\Big[\sum_{r=0}^{\infty} u_r(\eta_1)\zeta^r\Big]\Big[\sum_{r=0}^{\infty} \zeta^r\,\frac{(\Sigma_{i=1}^2 n_i(t_i-\eta_i))^r}{r!}\Big]$$

$$= \sum_{r=1}^{\infty} \zeta^r \int_{\eta_1}^{t_1} g(u_1,r)n_1 \frac{\left(n_1(t_1-u_1)+n_2(t_2-\eta_2)\right)^r}{(r-1)!}\,du_1$$

$$+ \sum_{r=0}^{\infty} g(t_1,r)\,\frac{\left(n_2(t_2-\eta_2)\right)^r}{r!}\,\zeta^r, \qquad \eta_1 < t_1,\ \zeta>0 \quad (2.3.89)$$

As before, write

$$v_r(\eta_1) = \sum_{j=0}^{r} u_j(\eta_1) \frac{\left(\Sigma_{i=1}^{2} n_i(t_i-\eta_i)\right)^{r-j}}{(r-j)!} \qquad r=0,1,2,\ldots \tag{2.3.90}$$

Equating coefficients on both sides of (2.3.89), one gets

$$v_0(\eta_1) = g(t_1,0) \tag{2.3.91}$$

$$v_r(\eta_1) = \int_{\eta_1}^{t_1} g(u_1,r) \frac{n_1\left(n_1(t_1-u_1)+n_2(t_2-\eta_2)\right)^{r-1}}{(r-1)!} du_1$$

$$+ g(t_1,r) \frac{\left(n_2(t_2-\eta_2)\right)^r}{r!} \qquad r\geq 1 \tag{2.3.92}$$

Differentiating (2.3.92) with respect to $\eta_1 < t_1$, and putting u_1 for η_1 we obtain

$$\frac{\partial v_r(u_1)}{\partial u_1} = \frac{-n_1\left(n_1(t_1-u_1) + n_2(t_2-\eta_2)\right)^{r-1}}{(r-1)!} g(u_1,r) \tag{2.3.93}$$

which implies

$$g(u_1,r) = -(r-1)!n_1^{-1}\left(n_1(t_1-u_1)+n_2(t_2-\eta_2)\right)^{-(r-1)} \frac{\partial v_r(u_1)}{\partial u_1},$$

$$\eta_1 < u_1 < t_1, \quad r \geq 1.$$

Also from (2.3.93)

$$g(t_1,r) = v_r(t_1)\left(\frac{\left(n_2(t_2-\eta_2)\right)^r}{r!}\right)^{-1} \qquad r=1,2,\ldots \tag{2.3.94}$$

Using all this information along with $v_0(\eta_1) = u_0(\eta_1) = g(t_1,0)$, one gets $Eg(U_1,R) = \sum_{r=0}^{\infty} u_r(\eta)\zeta^r$

Also, via completeness of U_1 and R and using the results thus far obtained in the theorem, it follows that only functions of u_1 and r which are continuous in $u_1 \varepsilon (0,t_1)$ can be unbiased estimators of differentiable (in η_1) parametric functions. □

Hence η_1 is not estimable but ζ is estimable and has UMVUE given by

$$g(U_1,R) = \frac{R-1}{n_1(t_1-U_1)+n_2(t_2-\eta_2)} I_{[R\geq 2,\eta_1<U_1<t_1]} + \frac{R}{n_2(t_2-\eta_2)} I_{[R\geq 1,\ U_1=t_1]}$$

Because of the "symmetry" of (2.3.77)-(2.3.80) with respect to u_1 and u_2, the case where η_1 is known but η_2 and ζ are unknown can be treated similarly.

Hence, if any function $h(\eta_2,\zeta)$ say is estimable then it is of the form $\sum_{r=0}^{\infty} u_r(\eta_2)\zeta^r$ where $u_0(\eta_2)$ does not depend on η_2, and this is also a sufficient condition when considering differentiable (in η_2) parametric functions. In such a case the only candidates for unbiased estimators are those that are continuous in $u_2\varepsilon(0,t_2)$.

The UMVUE of ζ in this case is given by

$$g(U_2,R) = \frac{R-1}{n_1(t_1-\eta_1)+n_2(t_2-U_2)} I_{[R\geq 2,\ \eta_2<U_2<t_2]} + \frac{R}{n_1(t_1-\eta_1)} I_{[R\geq 1,\ U_2=t_2]}$$

When both η_1 and η_2 are known, R is complete sufficient for ζ. Also R~Poisson $([n_1(t_1-\eta_1)+n_2(t_2-\eta_2)]\zeta)$. We already know from the similarity with the one sample case with known η, that for any function $h(\zeta)$ to have an unbiased estimator it is necessary and sufficient that it is of the form $\sum_{r=0}^{\infty} a_r\zeta^r$.

Hence from (2.2.18a) ζ is estimable with UMVUE given by

$$\tilde{\tilde{\zeta}} = \frac{R}{n_1(t_1-\eta_1)+n_2(t_2-\eta_2)}.$$

If ζ is known, U_1,U_2 are sufficient for (η_1,η_2) with pdf given by

$$f(u_1,u_2) = n_1 n_2 \zeta^2 \exp[-\zeta\Sigma_{i=1}^2 n_i(u_i-\eta_i)]\eta_1<u_1<t_1,\ \eta_2<u_2<t_2 \quad (2.3.95)$$

$$f(t_1,u_2) = n_2\zeta\exp[-\zeta(n_1(t_1-\eta_1)+n_2(u_2-\eta_2))] \quad \eta_2<u_2<t_2 \quad (2.3.96)$$

$$f(u_1,t_2) = n_1\zeta\exp[-\zeta(n_1(u_1-\eta_1)+n_2(t_2-\eta_2))] \quad \eta_1<u_1<t_1 \quad (2.3.97)$$

$$f(t_1,t_2) = \exp[-\zeta(n_1(t_1-\eta_1)+n_2(t_2-\eta_2))] \quad (2.3.98)$$

Using familiar arguments, it can be shown that this density is complete.

Furthermore if $Eg(U_1,U_2) = u(\eta_1,\eta_2)$ for some continuous function $g(u_1,u_2)$, then

$$u(\eta_1,\eta_2)\exp[-n_1\zeta\eta_1-n_2\zeta\eta_2] = \int_{\eta_2}^{t_2}\int_{\eta_1}^{t_1} g(u_1,u_2)$$

$$n_1 n_2 \zeta^2 \exp[-\zeta(n_1u_1+n_2u_2)]du_1du_2$$

$$+ \int_{\eta_1}^{t_1} g(u_1,t_2)n_1\zeta\exp[-\zeta(n_1u_1+n_2t_2)]du_1$$

$$+ \int_{\eta_2}^{t_2} g(t_1,u_2)n_2\zeta\exp[-\zeta(n_1t_1+n_2u_2)]du_2$$

$$+ g(t_1,t_2,0)\exp[-\zeta(n_1t_1+n_2t_2)] \quad (2.3.99)$$

Differentiating with respect to $\eta_2 < t_2$ and $\eta_1 < t_1$, and putting u_1 and u_2 for η_1 and η_2 respectively, one gets

$$\{[-\frac{\partial u(u_1,u_2)(n_1\zeta)}{\partial u_2} + \frac{\partial^2 u(u_1,u_2)}{\partial u_2 \partial u_1}]$$

$$+ [u(u_1,u_2)(-n_1\zeta) + \frac{\partial(u_1,u_2)}{\partial u_1}](-n_2\zeta)\} \exp[-\zeta(n_1u_1+n_2u_2)]$$

$$= g(u_1,u_2)n_1n_2\zeta^2\exp[-\zeta(n_1u_1+n_2u_2)]$$

which implies

$$g(u_1,u_2) = u(u_1,u_2) - \frac{\partial u(u_1,u_2)}{\partial u_2}\frac{1}{n_2\zeta} - \frac{\partial u(u_1,u_2)}{\partial u_1}\frac{1}{n_1\zeta} + \frac{\partial^2 u(u_1,u_2)}{\partial u_2\partial u_1}\frac{1}{n_1n_2\zeta^2} \qquad (2.3.100)$$

Similarly

$$g(u_1,t_2) = [u(u_1,t_2) - \frac{\partial u(u_1,t_2)}{\partial u_1}\cdot\frac{1}{n_1\zeta}] \qquad (2.3.101)$$

$$g(t_1,u_2) = [u(t_1,u_2) - \frac{\partial u(t_1,u_2)}{\partial u_2}\cdot\frac{1}{n_2\zeta}] \qquad (2.3.102)$$

$$g(t_1,t_2) = u(t_1,t_2). \qquad (2.3.103)$$

Hence if $u(\eta_1,\eta_2)$ is differentiable in $\eta_1\varepsilon(0,t_1)$ and $\eta_2\varepsilon(0,t_2)$, then $u(\eta_1,\eta_2)$ admits an unbiased estimator if and only if $u(t_1,t_2) = g(t_1,t_2)$. In such case only continuous functions of u_1 and u_2 can be unbiased for parametric functions $u(\eta_1,\eta_2)$ which are differentiable in η_1 and η_2. In particular both η_1 and η_2 have an unbiased estimator which is given by

$$\tilde{\eta}_i = U_i - (n_i\zeta)^{-1}[1 - I_{[U_i>t_i]}] \qquad i=1,2$$

and via RBLS, $\tilde{\eta}_i(i=1,2)$ is the UMVUE for $\eta_i(i=1,2.)$.

Finally, we consider the case when $\eta_1 = \eta_2 = \eta$ and $\zeta_1 = \zeta_2 = \zeta$. In this case, using (2.3.14)-(2.3.17), (Z,R) is sufficient for (η,ζ), where $Z = \min(X_{(11)}, X_{(21)})$ and $R = R_1+R_2$.

The joint pdf of (Z, R_1, R_2) is given by (2.3.23)-(2.3.27) and is denoted here by $q(z,r_1,r_2)$. Put $w = r_1$, then $r_2 = r-w$ and using (2.3.23)-(2.3.25) for $\eta < z < t_1$, $r > 1$, and $d = \Sigma_{i=1}^2 n_i(t_i-\eta)$, one gets

$$f(z,r) = \sum_{w=1}^{r} q(z,w,r-w)$$

$$= e^{-d\zeta}\Big[\sum_{w=1}^{r} \frac{(t_1-z)^{w-1}}{(w-1)!}\frac{(t_2-z)^{r-w}}{(r-w)!}(n_1\zeta)^w(n_2\zeta)^{r-w}$$

$$+ \sum_{w=0}^{r-1} \frac{(n_1\zeta)^w(n_2\zeta)^{r-w}(t_1-z)^w(t_2-z)^{r-1-w}}{w!\,(r-1-w)!}\Big]$$

$$= \frac{e^{-d\zeta}}{(r-1)!}\sum_{w=1}^{r} n_1\zeta\binom{r-1}{w-1}\big(n_1\zeta(t_1-z)\big)^{w-1}\big(n_2\zeta(t_2-z)\big)^{(r-1)-(w-1)}$$

$$+ \sum_{w=0}^{r-1} n_2\zeta\binom{r-1}{w}\big(n_1\zeta(t_1-z)\big)^{w}\big(n_2\zeta(t_2-z)\big)^{(r-1)-w}$$

$$= \frac{e^{-d\zeta}\zeta^r}{(r-1)!}\, n_1\big(n_1(t_1-z)+n_2(t_2-z)\big)^{r-1} + n_2\big(n_1(t_1-z)+n_2(t_2-z)\big)^{r-1}$$

$$= \frac{e^{-d\zeta}}{(r-1)!}(n_1+n_2)\zeta^r\big(n_1(t_1-z)+n_2(t_2-z)\big)^{r-1} \qquad (2.3.104)$$

for $r > 1$, $\eta < z < t_1$.

For $z = t_1$, using (2.3.26)-(2.3.27)

$$P(Z=t_1,\ R=0) = e^{-d\zeta} \qquad (2.3.105)$$

$$P(Z = t_1,\ R = r) = \frac{\big(n_2\zeta(t_2-t_1)\big)^r}{r!}e^{-d\zeta} \qquad r>1. \qquad (2.3.106)$$

It can also be shown that (Z,R) is complete sufficient for (η,ζ).

Using (2.3.104)-(2.3.106) it is easy to check that if a parametric function $h(\eta,\zeta)$ is estimable then it must be of the form $\sum_{r=0}^{\infty} u_r(\eta)\zeta^r$ where $u_0(\eta)$ does not depend on η. Our next result characterizes estimable functions within this class which are also differentiable in $\eta(< t)$.

Theorem 2.3.14 A parametric function $h(\eta,\zeta)$ which is differentiable in $\eta < t_1$, admits an unbiased estimator if and only if it is of the form $\sum_{r=0}^{\infty} u_r(\eta)\zeta^r$ where $u_0(\eta)$ does not depend on η.

Furthermomre, the class of functions of Z and R that are unbiased estimators of parametric functions $h(\eta,\zeta)$ which are differentiable in η, consists exactly of functions of z and r that are continuous in z for $z \in (0,t_1)$, $r \geq 1$.

Proof Necessity has already been established. To prove sufficiency, let $g(z,r)$ be a continuous function of $z\varepsilon(0,t_1)$ when $r \geq 1$. Then

$$\left[\sum_{r=0}^{\infty} u_r(\eta)\zeta^r\right]\left[\sum_{r=0}^{\infty} \frac{(\Sigma_{i=1}^{2} n_i(t_i-\eta))^r\zeta^r}{r!}\right]$$

$$= \sum_{r=1}^{\infty} \zeta^r \int_{\eta}^{t_1} g(z,r)(n_1+n_2)\, \frac{(n_1(t_1-z)+n_2(t_2-z))^{r-1}}{(r-1)!}\, dz$$

$$+ \sum_{r=0}^{\infty} g(t_1,r)\zeta^r\, \frac{(n_2(t_2-t_1))^r}{r!} \qquad (2.3.107)$$

for $\eta < t_1$, $\zeta > 0$.

As before, write

$$v_r(\eta) = \sum_{j=0}^{r} u_j(\eta)\frac{\left(\Sigma_{i=1}^2 n_i(t_i-\eta)\right)^{r-j}}{(r-j)!} \quad (2.3.108)$$

After equating coefficients on both sides of (2.3.107) and using (2.3.108) one gets

$$v_0(\eta) = v_0(t_1) = u_0(\eta) = g(t_1,0) \quad (2.3.109)$$

$$v_r(t_1) = g(t_1,r)\,\frac{\left(n_2(t_2-t_1)\right)^r}{r!} \qquad r \geq 1; \quad (2.3.110)$$

For $r \geq 1$, $\eta < z < t_1$

$$v_r(\eta) = \int_\eta^{t_1} g(z,r)(n_1+n_2)\,\frac{\left(n_1(t_1-z)+n_2(t_2-z)\right)^{r-1}}{(r-1)!}\,dz \quad (2.3.111)$$

Differentiating (2.3.111) with respect to $\eta < t_1$ and putting z for η, one gets

$$\frac{\partial v_r(z)}{\partial z} = -\,g(z,r)(n_1+n_2)\,\frac{\left(n_1(t_1-z) + n_2(t_2-z)\right)^{r-1}}{(r-1)!}$$

Using all this information it follows that

$$Eg(Z,R) = \sum_{r=0}^{\infty} u_r(\eta)\zeta^r.$$

Hence $g(Z,R)$ is unbiased for $\sum_{r=0}^{\infty} u_r(\eta)\zeta^r$ and by LSRB it is also UMVUE. Also using familiar arguments the only functions that can be unbiased estimators of differentiable parametric functions are those that are continuous in $z\varepsilon(0,t_1)$ for $r \geq 1$. □

Thus in this case η is not estimable but ζ is estimable with

UMVUE given by

$$\tilde{\zeta} = (R-1)\ \{\sum_{i=1}^{2} n_i(t_i-Z)\}^{-1} I_{[R > 1]}.$$

If ζ is known, only Z is complete sufficient with pdf given by

$$f(z) = (n_1+n_2)\zeta \exp[-(n_1+n_2)\zeta(z-\eta)], \quad \eta<z<t_1$$

$$f(t_1) = \exp\ [-(n_1+n_2)\zeta(t_1-\eta)].$$

Using the similarity with the one sample case when ζ is known it follows that any differentiable function $u(\eta)$ is estimable if and only if $u(t_1) = g(t_1)$ and in such case the UMVUE is given by

$$g(Z) = \left(u(Z) - u'(Z)\left((n_1+n_2)\zeta\right)^{-1}\right)[1-I_{[Z=t_1]}] + u(t_1)I_{[Z=t_1]}.$$

Again only continuous functions of Z can be unbiased estimators of $u(\eta)$ when $u(\eta)$ is differentiable.

If η is known, while ζ is unknown, then R is complete sufficient for ζ and has a Poisson distribution with mean $\sum_{i=1}^{2} n_i(t_i-\eta)\zeta$. We already know that only functions of the form $\sum_{r=0}^{\infty} a_r\zeta^r$ are estimable and in particular the UMVUE for ζ is given by

$$\approx{\zeta} = R[\sum_{i=1}^{2} n_i(t_i-\eta)]^{-1}.$$

CHAPTER THREE

MAXIMUM LIKELIHOOD ESTIMATION FOR THE WITH REPLACEMENT CASE

3.1 Introduction

In this chapter, we consider Maximum Likelihood Estimation (MLE) under Type I censoring with replacement. The one sample problem is considered in Section 3.2. The MLEs of the location parameter and the failure rate are obtained. The exact mean squared error (MSE) of the MLE of the location parameter is calculated. Also, in this section, a modified MLE is proposed and it is shown to dominate the MLE, from the MSE criterion, by achieving asymptotically about 50% risk reduction. In addition, the proposed modified MLE is shown to achieve asymptotically about 100% bias reduction over the MLE. Asymptotic distributions of the MLEs of the location and the scale parameters, as well as the asymptotic distributions of the modified MLE are also obtained in this section.

The two sample problem is considered in Section 3.3. Several cases are treated including those where the location and/or the scale parameters of the two populations are equal. As in the one sample case, modified MLEs of the location parameters are shown to achieve asymptotically 100% bias reduction and 50% MSE reduction. Asymptotic distributions are obtained for the MLEs as well as the modified MLEs of location and scale parameters.

3.2 Estimation in the One Sample Case

Suppose that n itmes are put to test, and the lifetimes of these items are iid with common pdf

$$f(x) = \zeta\exp[-\zeta(x-\eta)]I_{[x > \eta]}, \tag{3.2.1}$$

where $I_A = 1$ if A happens and $I_A = 0$, otherwise. The duration of the experiment is fixed, and is denoted by t. It is assumed that $\eta < t$, since otherwise there are no failures. An item which fails before the termination time is either replaced by another item, or is repaired and tested again. The replacement items have an exponential distribution with the same failure rate ζ but with location parameter 0. It is also assumed that the lifetimes of the original and replacement parts are mutually independent.

It follows that the joint pdf of failure times and R, the number of failures, is given by (cf (1.1.5))

$$f(x_{(1)},\ldots,x_{(r)},r) = (n\zeta)^r\exp[-n\zeta(t-\eta)]I_{[\eta<x_{(1)}<\ldots<x_{(r)}<t]} \qquad r=1,2,\ldots$$

$$f(x_{(1)},0) = \exp[-n\zeta(t-\eta)]I_{[x_{(1)}>t]}.$$

Note that the parameter space for (η,ζ) is $(0,t]\times(0,\infty)$.

From the previous chapter we know that $(X_{(1)},R)$ is minimal sufficient for (η,ζ) with pdf given by (2.2.1).

The MLEs of η and ζ are given respectively by

$$\begin{aligned}\hat{\eta} &= X_{(1)} \text{ if } \eta < X_{(1)} < t\\ &= t \text{ if } X_{(1)} > t,\end{aligned} \tag{3.2.2}$$

and $$\hat{\zeta} = \{R/(n(t-\hat{\eta}))\}I_{[\hat{\eta} < t]}. \tag{3.2.3}$$

First we consider estimation of η. Note that using (2.2.19)-(2.2.20) the MLE $\hat{\eta}$ has MSE given by

$$E(\hat{\eta}-\eta)^2 = E[(X_{(1)}-\eta)^2 I_{[\eta<X_{(1)}<t]}] + (t-\eta)^2 P(X_{(1)}>t)$$

$$= 2(n\zeta)^{-2}[1-\{1+n\zeta(t-\eta)\} \exp(-n\zeta(t-\eta))]. \qquad (3.2.4)$$

In order to motivate the modified MLE, first observe that when ζ is known, the UMVUE of η is given by $\hat{\eta}-(n\zeta)^{-1}I_{[\eta<X_{(1)}<t]}$ (see Section 2.2). When ζ is unknown, we substitute $\hat{\zeta}$ for ζ in the above UMVUE expression, and get the modified MLE $\hat{\hat{\eta}}$ of η as

$$\hat{\hat{\eta}} = \hat{\eta} - (n\hat{\zeta})^{-1} I_{[\eta<X_{(1)}<t]}. \qquad (3.2.5)$$

The next theorem provides the MSE for $\hat{\hat{\eta}}$.

Theorem 3.2.1

$$E(\hat{\hat{\eta}}-\eta)^2 = E(\hat{\eta}-\eta)^2 - (t-\eta)^2 E[\frac{R-1}{R(R+1)(R+2)} I_{[R>2]}]. \qquad (3.2.6)$$

Proof

$$E(\hat{\hat{\eta}}-\eta)^2 = E(\hat{\eta}-\eta)^2 - 2E[R^{-1}(X_{(1)}-\eta)(t-X_{(1)})I_{[\eta<X_{(1)}<t]}] + E[R^{-2}(t-X_{(1)})^2 I_{[\eta<X_{(1)}<t]}]. \qquad (3.2.7)$$

From (2.2.1) and the fact that $R\sim$Poisson $(n\zeta(t-\eta))$, the conditional pdf of $X_{(1)}$ given $R = r(>0)$ is given by

$$f(x_{(1)}|r) = r(t-x_{(1)})^{r-1}/(t-\eta)^r, \quad \eta< x_{(1)}<t. \qquad (3.2.8)$$

Hence, for $r > 0$,

$$E[(t-X_{(1)})^2 I_{[\eta<X_{(1)}<t]}|R=r]$$

$$= \int_\eta^t \{(t-x)^2 r(t-x)^{r-1}/(t-\eta)^r\}dx = (t-\eta)^2 r/(r+2); \qquad (3.2.9)$$

$$E[(t-X_{(1)})(X_{(1)}-\eta)I_{[\eta<X_{(1)}<t]}|R=r]$$

$$= \int_{\eta}^{t}(x-\eta)(t-x)r(t-x)^{r-1}(t-\eta)^{-r}dx$$

$$= r\int_{\eta}^{t}(x-\eta)(t-x)^{r}(t-\eta)^{-r}dx$$

$$= r(t-\eta)^2\int_0^1 z(1-z)^r dz = r(t-\eta)^2(r+1)^{-1}(r+2)^{-1}. \qquad (3.2.10)$$

From (3.2.7), (3.2.9) and (3.2.10), it follows that

$$E(\hat{\hat{\eta}}-\eta)^2 = E(\hat{\eta}-\eta)^2 + (t-\eta)^2\, E\left(\left[\frac{1}{R(R+2)} - \frac{2}{(R+1)(R+2)}\right]I_{[R>1]}\right)$$

$$= E(\hat{\eta}-\eta)^2 - (t-\eta)^2\, E\left(\left[\frac{R-1}{R(R+1)(R+2)}\right]I_{[R>1]}\right). \qquad (3.2.11)$$

This completes the proof of Theorem 3.2.1. □

Next we investigate the asymptotic behavior of the two MSE's $E(\hat{\eta}-\eta)^2$ and $E(\hat{\hat{\eta}}-\eta)^2$. More precisely, the following theorem is proved.

Theorem 3.2.2

(i) $n^2\, E(\hat{\eta}-\eta)^2 \to 2\zeta^{-2}$ as $n \to \infty$; (3.2.12)

(ii) $n^2\, E(\hat{\hat{\eta}}-\eta)^2 \to \zeta^{-2}$ as $n \to \infty$.

Proof (i) is an immediate consequence of (3.2.4). To prove (ii), note that $\frac{R}{n\zeta(t-\eta)} \overset{d}{=} \frac{\sum_{i=1}^{n} Y_i}{n\zeta(t-\eta)}$ where $Y_1,\ldots Y_n$ is a random sample of iid random variables whose pdf is Poisson with mean $\zeta(t-\eta)$. Hence by the weak law of large numbers

$$\frac{\sum_{i=1}^{n} Y_i}{n} \overset{P}{\to} EY_1 = \zeta(t-\eta)$$

Likewise,

$$\frac{n^2\,(R-1)}{R(R+1)(R+2)} = [\tfrac{R-1}{n}](\tfrac{R}{n} \cdot \tfrac{R+1}{n} \cdot \tfrac{R+2}{n})^{-1} \overset{P}{\to} \zeta(t-\zeta)[\zeta(t-\eta)]^{-3}$$

$$= (\zeta(t-\eta))^{-2}.$$

Moreover, $\{n^2(R-1)/R(R+1)(R+2)\}^{1+\delta} I_{[R>2]} \le \{n^2R^{-2}\}^{1+\delta} I_{[R>2]}$

for $\delta > 0$ and for every $0 < \varepsilon < 1$,

$E[n^{2+2\delta}R^{-(2+2\delta)}I_{[R>2]}]$

$$= E[n^{2+2\delta}R^{-(2+2\delta)}\{I_{[2<R<n\zeta(t-\eta)(1-\varepsilon)]} + I_{[R>n\zeta(t-\eta)(1-\varepsilon)]}\}]$$

$$\le n^{2+2\delta}2^{-(2+2\delta)}P(R<n\zeta(t-\eta)(1-\varepsilon)) + \{\zeta(t-\eta)(1-\varepsilon)\}^{-(2+2\delta)}$$

$$\le n^{2+2\delta}2^{-(2+2\delta)}P(|R-n\zeta(t-\eta)|>\varepsilon n\zeta(t-\eta))$$

$$+ \{\zeta(t-\eta)(1-\varepsilon)\}^{-(2+2\delta)}. \tag{3.2.13}$$

Using Markov's inequality

$$P(|R-n\zeta(t-\eta)|>\varepsilon n\zeta(t-\eta))$$

$$\le E|R-n\zeta(t-\eta)|^{8+4\delta}(\varepsilon n\zeta(t-\eta))^{-(8+4\delta)} \le Kn^{-(4+2\delta)}, \tag{3.2.14}$$

where in (3.2.14), we have made use of the following lemma.

<u>Lemma 3.2.1</u> (See Serfling (2.22)) If $Y_1,\ldots Y_n$ are iid with $E|Y_1|^{\nu}<\infty$ for some $\nu > 0$ then $E(\sum_{i=1}^{n}(Y_i - EY_i))^{\nu} \le Kn^{\nu/2}$ where K is a constant which does not depend on n.

Hence, combining (3.2.13) and (3.2.14), one gets

$$\sup_{n\ge 1} n^{2+2\delta}\, E|(R-1)/R(R+1)(R+2)|^{1+\delta} I_{[R>2]}$$

$$\le \sup_{n\ge 1} n^{2+2\delta} E|R^{-(2+2\delta)} I_{[R>2]}|$$

$$\leq \sup_{n\geq 1} [n^{2+2\delta}2^{-(2+2\delta)}Kn^{-(4+2\delta)} + (\zeta(t-\eta)(1-\varepsilon))^{-(2+2\delta)}] = O(1),$$

which implies that $[n^2(R-1)/R(R+1)(R+2)]I_{[R>2]}$ is uniformly integrable in n. This together with a.s. convergence, implies that

$$E(n^2(R-1)/R(R+1)(R+2))I_{[R>2]} \to (\zeta(t-\eta))^{-2} \tag{3.2.15}$$

as $n \to \infty$.

Hence, using (3.2.11), (3.2.15) and the first part of this theorem, one gets

$$n^2E(\hat{\hat{\eta}}-\eta) \to \zeta^{-2} \text{ as } n \to \infty$$

which completes the proof of (ii). □

It follows as a consequence of this theorem that asymptotically $\hat{\hat{\eta}}$ achieves 50% risk reduction than $\hat{\eta}$. Table 3.2.1, on the next page, shows the percentage risk improvement of $\hat{\hat{\eta}}$ over $\hat{\eta}$ for certain combinations of t when $\eta=0$ and $\zeta=1$.

It is also possible to obtain the exact bias of $\hat{\eta}$ and $\hat{\hat{\eta}}$. Simple calculations yield

$$E(\hat{\eta}-\eta) = (n\zeta)^{-1}[1-\{1+n\zeta(t-\eta)\} \exp(-n\zeta(t-\eta))]. \tag{3.2.16}$$

Also,

$$E(\hat{\hat{\eta}}-\eta) = E(\hat{\eta}-\eta) - E[R^{-1}(t-\hat{\eta}) \, I_{[\hat{\eta}<t]}]$$

$$= E(\hat{\eta}-\eta) - E[R^{-1}\textstyle\int_\eta^t(t-x)R(t-x)^{R-1}(t-\eta)^{-R}dx]$$

$$= E(\hat{\eta}-\eta) - (t-\eta)E[(R+1)^{-1}{}_{[R\geq 1]}]. \tag{3.2.17}$$

Thus, $nE(\hat{\eta}-\eta) \to \zeta^{-1}$ as $n \to \infty$ and $nE(\hat{\hat{\eta}}-\eta) \to \zeta^{-1}-\zeta^{-1} = 0$ as $n \to \infty$.

TABLE 3.2.1. MSE's OF $\hat{\eta}$ and $\hat{\hat{\eta}}$ AND PERCENTAGE RISK IMPROVEMENT (PRI) OF $\hat{\hat{\eta}}$ OVER $\hat{\eta}$

	n = 5			n = 10		
t	$n^2MSE(\hat{\eta})$	$n^2MSE(\hat{\hat{\eta}})$	PRI	$n^2MSE(\hat{\eta})$	$n^2MSE(\hat{\hat{\eta}})$	PRI
0.2	.5285	.5184	1.91	1.1880	1.1098	6.58
0.4	1.1880	1.0997	7.43	1.8168	1.4586	19.72
0.6	1.6017	1.3244	17.31	1.9653	1.3126	33.21
0.8	1.8168	1.2595	30.68	1.9940	1.1600	41.83
1.0	1.9191	1.0474	45.42	1.9990	1.0830	45.82
1.5	1.9906	.9070	54.44	2.0000	1.0788	46.06
1.8	1.9975	.7888	60.54	2.0000	1.0781	46.10

	n = 15			n = 20		
t	$n^2MSE(\hat{\eta})$	$n^2MSE(\hat{\hat{\eta}})$	PRI	$n^2MSE(\hat{\eta})$	$n^2MSE(\hat{\hat{\eta}})$	PRI
0.2	1.6017	1.3954	12.88	1.8168	1.4644	19.40
0.4	1.9653	1.3544	31.09	1.9940	1.2322	38.20
0.6	1.9975	1.1672	41.57	1.9998	1.1883	44.08
0.8	1.9998	1.1043	44.78	2.0000	1.1009	44.96
1.0	2.0000	1.0909	45.45	2.0000	1.0992	45.04
1.5	2.0000	1.0908	45.46	2.0000	1.0992	45.04
1.8	2.0000	1.0908	45.46	2.0000	1.0992	45.04

$$\text{PRI} = \frac{\text{MSE}(\hat{\eta}) - \text{MSE}(\hat{\hat{\eta}})}{\text{MSE}(\hat{\eta})} \times 100.$$

Hence, $\hat{\hat{\eta}}$ achieves asymptotically 100% bias reduction than $\hat{\eta}$, and, in (3.2.17), we have used the fact that

$$n(R+1)^{-1} I_{[R>1]} \to ((t-\eta)\zeta)^{-1} \overset{P}{\to} \text{ as } n \to \infty$$

and

$$\sup_{n>1} n^{2+2\delta} E\left((R+1)^{-1} I_{[R>1]}\right)^{2+2\delta} < \sup_{n>1} n^{2+2\delta} E\left(R^{-(2+2\delta)} I_{[R>1]}\right)$$

$$< \sup_{n>1} \left(n^{2+2\delta} K n^{-(3+2\delta)} + \left(\zeta(t-\eta)(1-\varepsilon)\right)^{-(2+2\delta)}\right) = 0(1)$$

which implies that $n(R+1)^{-1}$ is uniformly integrable in n.

Next we investigate the asymptotic distributions of $\hat{\eta}, \hat{\hat{\eta}}$ and $\hat{\zeta}$. Since, for all $\varepsilon > 0$,

$$P(\hat{\eta} \neq X_{(1)}) = P\left(X_{(1)} > t\right) = \exp\left(-n\zeta(t-\eta)\right),$$

and $\sum_{n=1}^{\infty} \exp\left(-n\zeta(t-\eta)\right) < \infty$, it follows from the Borel-Cantelli Lemma and the definition of $\hat{\eta}$ in (3.2.2) that $\hat{\eta} - X_{(1)} \xrightarrow{a.s.} 0$ as $n \to \infty$.

Moreover, for $0 < x < n(t_1-\eta)$,

$$\begin{aligned} P\left(n(X_{(1)}-\eta) > x\right) &= \int_{\frac{x+\eta}{n}}^{t} n\zeta \exp\left[-\zeta n\left(X_{(1)}-\eta\right)\right]dx \\ &\quad + \exp\left[-n\zeta(t-\eta)\right] \\ &= e^{-\zeta x} \qquad (3.2.18) \end{aligned}$$

and

$$P[n(X_{(1)}-\eta) = n(t_1-\eta)] = \exp[-n\zeta(t-\eta)] \tag{3.2.19}$$

Hence from (3.2.18) and (3.2.19) one gets $n(X_{(1)} - \eta)$ converges (as $n\to\infty$) to an exponential rv U with location parameter 0 and failure rate ζ. So, asymptotically, as $n\to\infty$

$$n(\hat{\eta}-\eta) \xrightarrow{d} U \tag{3.2.20}$$

The following theorem provides the asymptotic distribution of $n(\hat{\hat{\eta}}-\eta)$.

<u>Theorem 3.2.3</u> $n(\hat{\hat{\eta}}-\eta) \xrightarrow{d} U - \zeta^{-1}$ <u>as $n \to \infty$.</u> (3.2.21)

<u>Proof</u> Write

$$n(\hat{\hat{\eta}}-\eta) = n(\hat{\eta}-\eta) - \hat{\zeta}^{-1} I_{[\eta<X_{(1)}<t]}. \tag{3.2.22}$$

From the definition of $\hat{\zeta}$ in (3.2.3), $\hat{\zeta} \xrightarrow{P} \zeta$ as $n \to \infty$, also $I_{[\eta<X_{(1)}<t]} \xrightarrow{a.s.} 1$ as $n \to \infty$. The theorem now follows from

(3.2.20) and (3.2.22). □

Next we find the asymptotic distribution of $\hat{\zeta}$. It follows from (3.2.3) that

$$\sqrt{n}(\hat{\zeta}-\zeta) = [\sqrt{n}(\frac{R}{n(t-\eta)} - \zeta) + \frac{n(\hat{\eta}-\eta)}{(t-\hat{\eta})(t-\eta)} (R/n)n^{-1/2}]I_{[\hat{\eta}<t]}$$

$$+ \zeta\sqrt{n}(I_{[\hat{\eta}<t]}-1). \tag{3.2.23}$$

Since $R \sim$ Poisson $(n\zeta(t-\eta))$, $\sqrt{n}(\frac{R}{n(t-\eta)} - \zeta) \xrightarrow{L} N(0, \frac{\zeta}{t-\eta})$ by the C.L.T. and $R/n \xrightarrow{P} \zeta(t-\eta)$. Also, from (3.2.20), $n(\hat{\eta}-\eta) = O_p(1)$, so that $\hat{\eta} \xrightarrow{P} \eta$ (in fact, one can directly show

that $\hat{\eta} \xrightarrow{a.s.} \eta$). Thus, from (3.2.23),

$$\sqrt{n}(\hat{\zeta}-\zeta) \xrightarrow{L} N(0,\zeta(t-\eta)^{-1}).$$

Now, using the lemma that follows, one gets

$$\sqrt{n}(\hat{\zeta}^{-1} - \zeta^{-1}) \xrightarrow{L} N(0,\zeta^{-3}(t-\eta)^{-1}) \tag{3.2.24}$$

Lemma 3.2.2. Suppose that X_n is $AN(u,\sigma_n^2)$, with $\sigma_n \to 0$ as $n \to \infty$. Let g be a real valued function differentiable at $x = u$, with $g'(u) \neq 0$. Then

$$g(X_n) \text{ is } AN(g(u), [g'(u)]^2\sigma_n^2).$$

We shall next show that

$$[\sqrt{n}\ (\hat{\zeta}^{-1} - \zeta^{-1})]^2 \text{ is uniformly integrable in } n\geq 1, \tag{3.2.25}$$

so that from (3.2.24) and (3.2.25), one gets

$$E[n(\hat{\zeta}^{-1}- \zeta^{-1})^2] \to \zeta^{-3}(t-\eta)^{-1}. \tag{3.2.26}$$

In order to prove (3.2.25), first write

$$n^2(\hat{\zeta}^{-1}- \zeta^{-1})^4 = n^2[\frac{n(t-\hat{\eta})}{R} - \zeta^{-1}I_{[\hat{\eta}<t]}$$

$$+ \zeta^{-1}(I_{[\hat{\eta}<t]}-1)]^4$$

$$= n^2[\frac{n(\eta-\hat{\eta})}{R} + \frac{n(t-\eta)}{R} - \zeta^{-1}I_{[\hat{\eta}<t]}$$

$$+ \zeta^{-1}(I_{[\hat{\eta}<t]}-1)]^4$$

$$\leq 3^3n^2[\frac{n^4(\hat{\eta}-\eta)^4}{R^4}\ I_{[R>1]}+ |\frac{n(t-\eta)}{R} - \zeta^{-1}|^4I_{[R>1]}$$

$$+ \zeta^{-4}|I_{[\hat{\eta} < t]}-1|^4] \tag{3.2.27}$$

since $I_{[\hat{\eta} < t]} = I_{[R > 1]}$.

Note that, using the Schwarz inequality

$$E[(\hat{\eta}-\eta)^4 R^{-4} n^4 I_{[R>1]}] \leq E^{1/2}(\hat{\eta}-\eta)^8 \, E^{1/2}(R^{-8} n^8 I_{[R>1]})$$

where

$$E(\hat{\eta}-\eta)^8 = \int_\eta^t n\zeta(x_{(1)}-\eta)^8 e^{-n\zeta(x_{(1)}-\eta)} dx_{(1)} + (t-\eta)^8 e^{-n\zeta(t-\eta)}$$

$$= (n\zeta)^{-8}[\int_0^{n\zeta(t-\eta)} z^8 e^{-z} dz + (n\zeta(t-\eta))\, e^{-n\zeta(t-\eta)}]$$

$$\leq (n\zeta)^{-8}\, \Gamma[7]$$

and $E(R^{-8} n^8 I_{[R>1]}) = O(1)$ using arguments similar to (3.2.13) and (3.2.14). Hence

$$n^2 E[(\hat{\eta}-\eta)^4 R^{-4} n^4 I_{[R>1]}] = O(n^2 \cdot n^{-4}) = O(n^{-2})$$

which shows that

$$\sup_{n>1} E(n^3(\hat{\eta}-\eta)^2 R^{-2} I_{[R>1]})^2 < \infty. \tag{3.2.28}$$

Moreover,

$$E[\{\frac{n(t-\eta)}{R} - \zeta^{-1}\}^4 I_{[R>1]}]$$

$$= E[\{R - n\zeta(t-\eta)\}^4 R^{-4} \zeta^{-4} I_{[R>1]}]$$

$$\leq \zeta^{-4} E^{1/2}(R-n\zeta(t-\eta))^8 \, E^{1/2}[R^{-8} I_{[R>1]}]. \tag{3.2.29}$$

It is easy to check using Lemma 3.2.1, that

$E(R-n\zeta(t-\eta))^8 = O(n^4)$ while $E[R^{-8} I_{[R>1]}] \leq$

$P(R<n\zeta(t-\eta)(1-\varepsilon)) + (n\zeta(t-\eta)(1-\varepsilon))^{-8}$

$\leq P(|R-n\zeta(t-\eta)| > n\varepsilon\zeta(t-\eta)) + (n\zeta(t-\eta)(1-\varepsilon))^{-8} = O(n^{-8}).$

Hence, from (3.2.29), it follows that

$$n^2E[\{\frac{n(t-\eta)}{R} - \zeta^{-1}\}^4 I_{[R>1]}] = O(n^2n^2n^{-4}) = O(1). \tag{3.2.30}$$

Also,

$$n^2E\zeta^{-4}(I_{[\hat{\eta}<t]}-1)^4$$

$$= n^2\zeta^{-4} P[\hat{\eta} > t] = n^2\zeta^{-4}e^{-n\zeta(t-\eta)} = o(1) \tag{3.2.31}$$

Now, it follows from (3.2.27),(3.2.28),(3.2.30) and (3.2.31) that

$$\sup_{n>1} E[n^2(\hat{\zeta}^{-1} -\zeta^{-1})^4] < \infty. \tag{3.2.32}$$

Note that (3.2.25) is an immediate consequence of (3.2.32).

3.3 Estimation in the Two Sample Case

Suppose that two independent sets of items are put to test, where the first set contains n_1 elements, and the second set contains n_2 elements. The lifetimes of the items in the i^{th} set are assumed to be iid with common pdf

$$f(x) = \zeta_i\exp[-\zeta_i(x-\eta_i)]I_{[x>\eta_i]} \quad (i=1,2). \tag{3.3.1}$$

Once again, the duration of the experiment is fixed, and the censoring times for the two sets are denoted by t_1 and t_2. We assume that $\eta_i < t_i (i=1,2)$. Also, for definiteness let $t_1 < t_2$. An item which fails before the termination time is either replaced by another item, or is repaired and tested again. The replacement items from set i have an exponential distribution with the same failure rate ζ_i but with location parameter $0 (i=1,2)$. We denote by R_i the number of failures before time t_i for the set $i (i=1,2)$. Then R_i's are independent with $R_i \sim$ Poisson $(n_i\zeta_i(t_i-\eta_i))$, $i=1,2$.

Given $R_i = r_i(>0)$, the order statistics for set i are denoted by $X_{(i1)} <,\ldots,< X_{(ir_i)} (i=1,2)$. First consider the case when η_1, η_2, ζ_1 and ζ_2 are all distinct and unknown. In this case, the MLEs of η_i's and ζ_i's are given respectively by

$\hat{\eta}_i = X_{(i1)} I_{[X_{(i1)} < t_i]} + t_i I_{[X_{(i1)} > t_i]}$ and

$\hat{\zeta}_i = \{R_i/(n_i(t_i - \hat{\eta}))\} \; I_{[\hat{\eta}_i < t_i]}$ i=1,2. Using a straightforward extension of the results of Section 3.2, it follows

that $\hat{\hat{\eta}}_i = \hat{\eta}_i - (n_i \hat{\zeta}_i)^{-1} \; I_{[\eta_i < X_{(i1)} < t_i]}$ will achieve asymptotically 100% bias reduction and 50% MSE reduction than $\hat{\eta}_i$ (i=1,2).

Also the scale parameter,

$$\hat{\zeta}_i^{-1} \sim AN\left(\zeta_i^{-1}, \frac{\zeta_i^{-3}(t_i - \eta_i)^{-1}}{n_i}\right), \; (i=1,2)$$

and $n_i(\hat{\zeta}_i^{-1} - \zeta_i^{-1})^2$, is uniformily integrable as $n_i \to \infty$ which implies

$$E[n_i(\hat{\zeta}_i^{-1} - \zeta_i^{-1})^2] \to \zeta_i^{-3}(t_i - \eta_i)^{-1} \text{ and}$$

$$E[\sqrt{n_i}(\hat{\zeta}_i^{-1} - \zeta_i^{-1})] \to 0 \quad \text{ as } n_i \to \infty \; (i=1,2).$$

Next we consider the case when $\eta_1 = \eta_2 = \eta$, but ζ_1 and ζ_2 are not necessarily equal. First consider the case when η, ζ_1 and ζ_2 are all unknown. In this set up, estimation of η in the uncensored case was considered by Ghosh and Razmpour (1984), and for the type II censored case by Chiou and Cohen (1984).

Write $Z = \min\left(X_{(11)}, X_{(21)}\right)$. An examination of (2.3.14) - (2.3.17) of Chapter Two reveals that the MLE of η, ζ_1 and ζ_2 are given respectively by

$$\hat{\eta} = Z I_{[\eta<Z<t_1]} + t_1 I_{[Z>t_1]}, \quad \hat{\zeta}_i = \{R_i/n_i(t_i-\hat{\eta})\}\, I_{[\hat{\eta}<t_1]}.$$

The pdf of $\hat{\eta}$ is given by (2.3.66) and (2.3.67), where it was derived for known ζ_1 and ζ_2, but of course, remains unchanged when ζ_1 and ζ_2 are both unknown. Hence, once again

$$f(z) = a\exp\left(-a(z-\eta)\right), \quad \eta < z < t_1;$$

$$P(Z = t_1) = \exp\left(-a(t_1-\eta)\right),$$

where $a = n_1\zeta_1 + n_2\zeta_2$.

Note, from above that $\hat{\eta} \xrightarrow{a.s.} \eta$ as $\min(n_1,n_2) \to \infty$ since $\sum_{n=1}^{\infty} P\left[|\hat{\eta} - \eta| > \varepsilon\right] = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} \exp\left[-a(t_1-\eta)\right] < \infty$. Next, to find the asymptotic distribution of $\hat{\eta}$, first let $n = n_1 + n_2$. Assume that

$$\lim_{n\to\infty} n_1/n = \lambda, \quad 0 < \lambda < 1. \tag{3.3.2}$$

The next theorem provides the asymptotic distribution of $\hat{\eta}$.

<u>Theorem 3.3.1</u> <u>Assume</u> (3.3.2). <u>Then</u> $n(\hat{\eta} - \eta)$ <u>converges in distribution to an exponential random variable with failure rate</u> $\lambda\zeta_1 + (1-\lambda)\zeta_2$.

<u>Proof</u> Note that for every u in $\left(0, n(t_1-\eta)\right)$,

$$P\left(n(\hat{\eta}-\eta)<u\right) = 1-\exp(-au/n) \to 1-\exp\left(-(\lambda\zeta_1 + (1-\lambda)\zeta_2)u\right), \tag{3.3.3}$$

as $n \to \infty$. Also

$$P(\hat{\eta} = t_1) = \exp\left(-a(t_1-\eta)\right) \to 0 \text{ as } n \to \infty. \tag{3.3.4}$$

This proves the theorem. □

When the two populations have common location parameter η, if ζ_1 and ζ_2 are known, the UMVUE of η was obtained in Chapter Two and is given by $\hat{\eta} - a^{-1}I_{[\eta<Z<t_1]}$. Thus, when ζ_1 and ζ_2 are unknown, we propose the modified ML estimator of η as

$$\hat{\hat{\eta}} = \hat{\eta} - \hat{a}^{-1}I_{[\eta<Z<t_1]}, \tag{3.3.5}$$

where $\hat{a}$ is obtained by plugging the ML estimators of $\hat{\zeta}_1$ and $\hat{\zeta}_2$ for ζ_1 and ζ_2 in a.

Note $\hat{\zeta}_i = \dfrac{R_i}{n_i(t_i-\hat{\eta})} I_{[\eta<\hat{\eta}<t_1]} = \dfrac{R_i}{n_i(t_i-\eta)} \cdot \dfrac{(t_i-\eta)}{(t_i-\hat{\eta})} I_{[\eta<\hat{\eta}<t_1]}$

for i=1,2.

Recall $\dfrac{R_i}{n_i(t_i-\eta)} \overset{P}{\to} \zeta_i$ and $\hat{\eta} \xrightarrow{a.s.} \eta$ as min $(n_1,n_2) \to \infty$.

Also, $I_{[\eta<Z<t_1]} \xrightarrow{a.s.} 1$ since

$$\sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} P[|I_{[\eta<Z<t_1]}-1| > \varepsilon] = \sum_{n_1=1}^{\infty} \sum_{n_2=1}^{\infty} e^{-a(t_1-\eta)} < \infty.$$

Hence $\hat{\zeta}_i \overset{P}{\to} \zeta_i$ as $\min_{i=1,2} n_i \to \infty$ for i = 1,2 and under the assumption (3.3.2)

$$n\hat{a}^{-1} = \left(\frac{n_1\hat{\zeta}_1}{n} + \frac{n_2\hat{\zeta}_2}{n}\right)^{-1} \xrightarrow{a.s.} (\lambda\zeta_1 + (1-\lambda)\zeta_2)^{-1} = g^{-1}(\text{say}).$$

Hence using Theorem 3.3.1 $n(\hat{\hat{\eta}} - \eta)$ converges in distribution to W $- g^{-1}$, where W is exponential with failure rate g. Next we find asymptotic MSE's of $\hat{\eta}$ and $\hat{\hat{\eta}}$. Direct calculations give

$$E[n^2(\hat{\eta}-\eta)^2] = (2n^2/a^2)[1-(1+a(t_1-\eta))\exp(-a(t_1-\eta))]$$
$$+ 2(\lambda\zeta_1 + (1-\lambda)\zeta_2)^{-2} = 2g^{-2} \qquad (3.3.6)$$

as $n \to \infty$ using (3.3.2) and the fact that $a = n_1\zeta_1 + n_2\zeta_2$. We next show that

$$E[n^2(\hat{\hat{\eta}}-\eta)^2] \to g^{-2} \text{ as } n \to \infty. \qquad (3.3.7)$$

Since we have already shown that $n(\hat{\hat{\eta}}-\eta)$ converges in distribution to $W-g^{-1}$, where W is exponential with failure rate g^{-1}, it remains only to prove that

$$\{n^2(\hat{\hat{\eta}}-\eta)^2\} \text{ is uniformly integrable (u.i) in } n \geq 1. \qquad (3.3.8)$$

In order to prove (3.3.8), first use the inequality

$$n^2(\hat{\hat{\eta}}-\eta)^2 \leq 2[n^2(\hat{\eta}-\eta)^2 + n^2\hat{a}^{-2}I_{[\eta<Z<t_1]}]. \qquad (3.3.9)$$

The u.i. property of $n^2(\hat{\eta}-\eta)^2$ follows easily by showing that

$$E(n^4(\hat{\eta}-\eta)^4) = \int_\eta^t a(n(x-\eta))^4\exp[-a(x-\eta)]dx$$
$$+ n^4(t_1-\eta)^4\exp[-a(t_1-\eta)]$$
$$\leq n^4a^{-4}\int_0^\infty z^4\exp[-z]dz + n^4(t_1-\eta)^4\exp[-a(t_1-\eta)]$$
$$= n^4a^{-4}\Gamma(3) + n^4(t_1-\eta)^4\exp[-a(t_1-\eta)]$$
$$=O(1). \qquad (3.3.10)$$

Thus, we need only prove

Theorem 3.3.2 $n^2\hat{a}^{-2}I_{[\eta<Z<t_1]}$ is u.i. in n.

Proof The proof follows by showing that

$$\sup_{n\geq 1} E[n^3\hat{a}^{-3}I_{[\eta<Z<t_1]}] < \infty. \qquad (3.3.11)$$

But,

$$n^3E[\hat{a}^{-3}I_{[\eta<Z<t_1]}]$$

$$= n^3E[(n_1\hat{\zeta}_1 + n_2\hat{\zeta}_2)^{-3}(I_{[R_1\geq 1]}+I_{[R_1=0,R_2\geq 1]})I_{[\eta<Z<t_1]}$$

$$\le n^3[(n_1\hat{\zeta}_1)^{-3}I_{[R_1>1]} + (n_2\hat{\zeta}_2)^{-3}I_{[R_2>1]}]$$

$$= n^3E[\Sigma_{i=1}^2(t_i-\hat{\eta})^3R_i^{-3}I_{[R_i>1]}]$$

$$\le n^3\ \Sigma_{i=1}^2(t_i-\eta)^3E[R_i^{-3}I_{[R_i>1]}]. \qquad (3.3.12)$$

Arguments similar to (3.2.13) will now show that the right hand side of (3.3.12) is $O(1)$. □

Simple calculations also show that $E(n(\hat{\eta}-\eta)) \to g^{-1}$. On the other hand, since $n^2(\hat{\hat{\eta}}-\eta)^2$ is uniformly integrable (u.i.) then $n(\hat{\hat{\eta}}-\eta)$ is also u.i., which together with the fact that $n(\hat{\hat{\eta}}-\eta) \xrightarrow{L} W-g^{-1}$, implies that

$En(\hat{\hat{\eta}}-\eta) \to E(W-g^{-1}) = 0$. Thus, $\hat{\hat{\eta}}$ achieves asymptotically 100% bias reduction than $\hat{\eta}$.

Next, using arguments similar to the ones given in Section 3.2 one gets

$\sqrt{n_i}\ (\hat{\zeta}_i-\zeta_i) \xrightarrow{L} N(0,\zeta_i/(t_i-\eta))$ as $\min(n_1,n_2) \to \infty$ $(i=1,2)$.

Using Lemma 3.2.2, it follows now that

$$\sqrt{n_i}\ (\hat{\zeta}_i^{-1}-\zeta_i^{-1}) \xrightarrow{L} N(0,\zeta_i^{-3}(t_i-\eta)^{-1})\ (i=1,2).$$

Calculations similiar to (3.2.26)-(3.2.32) will then yield

$$E[n_i(\hat{\zeta}_i^{-1}-\zeta_i^{-1})^2] \to \zeta_i^{-3}(t_i-\eta)^{-1} \text{ as } \min(n_1,n_2) \to \infty\ (i=1,2).$$

Next, we consider the case when $\zeta_1 = \zeta_2 = \zeta$, but η_1 and η_2 are not necessarily equal. In this case, using (1.1.9) the joint pdf of $X_{(11)},\ldots,X_{(1R_1)}$, $X_{(21)}\ldots X_{(2R_2)}$, R_1 and R_2 is given by

$f(x_{(11)},\ldots,x_{(1r_1)},x_{(21)},\ldots x_{(2r_2)},r_1,r_2)$

$$= \left(\prod_{i=1}^{2} n_i^{r_i}\right)\zeta^{r_1+r_2}\exp[-\zeta\Sigma_{i=1}^2 n_i(t_i-\eta_i)]I_{[\eta_1<x_{(11)}<\ldots x_{(1r_1)}<t_1]}$$

$$\times I_{[\eta_2<x_{(21)}<\ldots<x_{(2r_2)}<t_2]},\quad r_1>0, r_2>0; \qquad (3.3.13)$$

$f(x_{(21)},\ldots x_{(2r_2)},0,r_2)$

$$= n_2^{r_2}\zeta^{r_2}\exp[-\zeta\Sigma_{i=1}^2 n_i(t_i-\eta_i)]\times I_{[\eta_2<X_{(21)}<\ldots<X_{(2r_2)}<t_2]},$$

$$r_2>0; \qquad (3.3.14)$$

$f(x_{(11)},\ldots,x_{(1r_1)},\ r_1,\ 0)$

$$= n_1^{r_1}\zeta^{r_1}\exp[-\zeta\Sigma_{i=1}^2 n_i(t_i-\eta_i)]I_{[\eta_1<x_{(11)}<\ldots<x_{(1r_1)}<t_1]}$$

$$r_1>0; \qquad (3.3.15)$$

$$f(0,0) = \exp\left[-\zeta\Sigma_{i=1}^2 n_i(t_i-\eta_i)\right]. \qquad (3.3.16)$$

From (3.3.13)-(3.3.16), it follows that writing $R = R_1 + R_2$, $(X_{(11)}, X_{(21)}, R)$ is sufficient for (η_1,η_2,ζ) and the MLEs of η_1,η_2 and ζ are given respectively by

$\hat{\eta}_i = X_{(i1)}I_{[\eta_i<X_{(i1)}<t_i]} + I_{[X_{(i1)}>t_i]}$ and

$$\hat{\zeta} = R/\{\Sigma_{i=1}^{2} n_i(t_i-\hat{\eta}_i)\} I_{[R>1]}. \qquad (3.3.17)$$

Since $X_{(i1)}$ (i=1,2) has pdf

$$f(x_{(i1)}) = n_i\zeta \exp[-n_i\zeta(x_{(i1)}-\eta_i)] \qquad \eta_i<x_{(i1)}<t_i$$
$$f(t_i) = \exp[-n_i\zeta(t_i-\eta_i)] \qquad (3.3.18)$$

then, arguing as in previous cases $n_i(X_{(i1)}-\eta_i)$ converges in distribution to an exponential random variable with failure rate ζ.

Also, if (3.3.2) holds, then, we write

$$\sqrt{n}(\hat{\zeta}-\zeta) = \sqrt{n}\Big(\frac{R}{n_1(t_1-\eta_1) + n_2(t_2-\eta_2)} - \zeta\Big) I_{[R>1]}$$

$$+ \sqrt{n}\, R\, \frac{(n_1(X_{(11)}-\eta_1) + n_2(X_{(21)}-\eta_2)) I_{(R>1]}}{(n_1(t_1-\eta_1) + n_2(t_2-\eta_2))(n_1(t_1-X_{(11)}) + n_2(t_2-X_{(21)}))}]$$

$$+ \sqrt{n}\zeta(I_{[R>1]}-1)$$

$$= \sqrt{n}\Big(\frac{R}{n_1(t_1-\eta_1) + n_2(t_2-\eta_2)} - \zeta\Big) I_{[R>1]}$$

$$+ n^{-1/2}\frac{R}{n}\, \frac{(n_1(X_{(11)}-\eta_1) + n_2(X_{(21)}-\eta_2)) I_{[R>1]}}{[\frac{n_1}{n}(t_1-\eta_1) + \frac{n_2}{n}(t_2-\eta_2)][\frac{n_1}{n}(t_1-X_{(11)}) + \frac{n_2}{n}(t_2-X_{(21)})]}$$

$$+\sqrt{n}\ \zeta(I_{R>1]} - 1) \qquad (3.3.19)$$

Next, we give the limiting behavior of each of the three terms in (3.3.19). We rewrite the first term as follows:

$$\sqrt{n}\Big(\frac{R}{n_1(t_1-\eta_1)+n_2(t_2-\eta_2)} - \zeta\Big) I_{[R>1]}$$

$$= \sqrt{n}\Big(\frac{R}{n_1+n_2} - \frac{(n_1(t_1-\eta_1)+n_2(t_2-\eta_2))\zeta}{n_1+n_2}\Big\}$$

$$\times \frac{n_1 + n_2}{n_1(t_1-\eta_1) + n_2(t_2-\eta_2)} I_{[R>1]} \tag{3.3.19a}$$

In order to obtain the limiting distribution of the above term we make use of the following lemma.

Lemma 3.3.1 If $\{X_n, \eta > 1\}$ are independent random variables with $EX_n = 0, EX_n^2 = \sigma_n^2$, $E|X_n|^{2+\delta} < \infty$ for some $\delta > 0$, all $n > 1$ and if $\sum_{j=1}^{n} E|X_j|^{2+\delta} = o(s_n^{2+\delta})$ where, $s_n^2 = \sum_{j=1}^{n} \sigma_j^2$, $n>1$, then

$$\frac{\sum_{j=1}^{n} X_j}{s_n} \xrightarrow{d} N(0,1).$$

Hence, let $\delta = 2$. As before $n = n_1 + n_2$.

$$\text{Define } X_j = \begin{cases} W_j-(t_1-\eta_1)\zeta & j=1,2,\ldots,n_1 \\ Z_j-(t_2-\eta_2)\zeta & j=n_1+1,\ldots,n \end{cases}$$

where the W_j's are i.i.d. Poisson variables with mean $(t_1-\eta_1)\zeta$ and the Z_j's are i.i.d. Poisson variables with mean $(t_2-\eta_2)\zeta$.

Then

$$EX_j = \begin{cases} EW_j - (t_1-\eta_1)\zeta = 0 & j=1,2,\ldots,n_1, \\ EZ_j - (t_2-\eta_2)\zeta = 0 & j=n_1+1,\ldots,n \end{cases}$$

$$\sigma_j^2 = \begin{cases} (t_1-\eta_1)\zeta & j=1,2,\ldots,n_1 \\ (t_2-\eta_2)\zeta & j=n_1+1,\ldots,n \end{cases}$$

and

$$E|X_j|^4 = \begin{cases} (t_1-\eta_1)\zeta + 3(t_1-\eta_1)^2\zeta^2 & j=1,2,\ldots,n_1 \\ (t_2-\eta_2)\zeta + 3(t_2-\eta_2)^2\zeta^2 & j=n_1+1,\ldots,n \end{cases}$$

so that

$$\frac{\sum_{j=1}^{n} E|X_j|^4}{s_n^4} = \frac{n_1(t_1-\eta_1)\zeta+3n_1(t_1-\eta_1)^2\zeta^2+n_2(t_2-\eta_2)\zeta+3n_2(t_2-\eta_2)^2\zeta^2}{\left(n_1(t_1-\eta_1)\zeta + n_2(t_2-\eta_2)\zeta\right)^2}$$

$$\to 0 \qquad \text{as } \min(n_1,n_2) \to \infty.$$

Hence

$$\frac{\sum_{j=1}^{n} X_j}{s_n} = \frac{\sum_{j=1}^{n_1}\left(W_j-(t_1-\eta_1)\zeta\right) + \sum_{j=n_1+1}^{n}\left(Z_j-(t_2-\eta_2)\zeta\right)}{\sqrt{\left(n_1(t_1-\eta_1) + n_2(t_2-\eta_2)\right)\zeta}}$$

$$\stackrel{d}{=} \frac{\frac{R}{n} - \frac{\left(n_1(t_1-\eta_1) + n_2(t_2-\eta_2)\right)\zeta}{n}}{\sqrt{\frac{\left(n_1(t_1-\eta_1) + n_2(t_2-\eta_2)\right)\zeta}{n^2}}} \xrightarrow{d} N(0,1)$$

i.e.

$$\sqrt{n}\left(\frac{R}{n} - \frac{\zeta\left(n_1(t_1-\eta_1) + n_2(t_2-\eta_2)\right)}{n}\right) \sim AN\left\{0, \left(\lambda(t_1-\eta_1)+(1-\lambda)(t_2-\eta_2)\right)\zeta\right\}.$$

Hence it follows via Slutsky's theorem and the fact that $I_{[R>1]} \xrightarrow{P} 1$ that (3.3.19a) goes in distribution to a normal variable with mean zero and variance

$\zeta(\lambda(t_1-\eta_1) + (1-\lambda)(t_2-\eta_2))^{-1}$.

Next we note that the second term in (3.3.19), namely

$$n^{-1/2}\,\frac{R}{n}\,\frac{(n_1\,(X_{(11)}-\eta_1) + n_2\,(X_{(21)}-\eta_2))I_{[R>1]}}{[\frac{n_1}{n}(t_1-\eta_1)+\frac{n_2}{n}(t_2-\eta_2)][\frac{n_1}{n}(t_1-X_{(11)})+\frac{n_2}{n}(t_2-X_{(21)})]} \qquad (3.3.20)$$

$= o_p(1)$.

Since, using (3.3.2) one gets that

$$\frac{R}{n} = \frac{n_1}{n}\frac{R_1}{n_1} + \frac{n_2}{n}\frac{R_2}{n_2} \xrightarrow{P} \lambda\zeta(t_1-\eta_1) + (1-\lambda)\zeta(t_2-\eta_2).$$

Also $X_{(i1)} \xrightarrow{P} \eta_i$ for $i=1,2$ and $I_{[R>1]} \xrightarrow{P} 1$

as $\min\limits_{i=1,2} n_i \to \infty$ and $\sum\limits_{i=1}^{2} n_i(X_{(i1)} - \eta_i)$ is $O_p(1)$.

Hence the result in (3.3.20).

Finally, using the Borel-Cantelli lemma,

$\sqrt{n}\zeta(I_{[R>1]}-1) \xrightarrow{a.s.} 0$

as $\min(n_1,n_2) \to \infty$ since

$$\sum_{n=1}^{\infty} P(\sqrt{n}\;\zeta(I_{[R>1]}-1) \neq 0) =$$

$$\sum_{n=1}^{\infty} P(I_{[R>1]}\neq 1) = \sum_{n=1}^{\infty} P[R=0] = \sum_{n_1=1}^{\infty}\sum_{n_2=1}^{\infty} \exp[-\zeta(n_1(t_1-\eta_1)$$

$$+ n_2(t_2-\eta_2)] < \infty. \qquad (3.2.21)$$

Hence, combining (3.3.18) through (3.3.21) one gets that

$$\sqrt{n}\,(\hat{\zeta}-\zeta) \xrightarrow{L} N\{0,\ \zeta(\lambda(t_1-\eta_1) + (1-\lambda)(t_2-\eta_2))^{-1}\}$$

For fixed ζ, since the UMVUE of η_i is

$X_{(i1)}-(n_i\zeta)^{-1}I_{[\eta_i<X_{(i1)}<t_i]}(i=1,2,)$, the modified MLE of η_i is given by

$$\hat{\hat{\eta}}_i = X_{(i1)} - (n_i\hat{\zeta})^{-1}I_{[\eta_i<X_{(i1)}<t_i]} \qquad (i=1,2).$$

In this case, asymptotically as $n_i \to \infty$

$n_i(\hat{\eta}_i-\eta_i) = n_i(X_{(i1)}-\eta_i) \xrightarrow{L} U$, ,an exponential random variable with failure rate ζ, while

$$n_i(\hat{\hat{\eta}}_i-\eta_i) = n_i(X_{(i1)}-\eta_i) - \hat{\zeta}^{-1}I_{[\eta_i<X_{(i1)}<t_i]} \xrightarrow{L} U-\zeta^{-1}$$

$$\text{since } \hat{\zeta} = \frac{R}{\sum_{i=1}^{2} n_i(t_i-\hat{\eta}_i)}\, I_{[R>1]} = \frac{\left[\frac{n_1}{n}\frac{R_1}{n_1} + \frac{n_2}{n}\frac{R_2}{n_2}\right]}{\frac{n_1}{n}(t_1-\eta_1) + \frac{n_2}{n}(t_2-\eta_2)} \cdot I_{[R>1]}$$

$$\xrightarrow{P} \frac{\zeta(\lambda(t_1-\eta_1) + (1-\lambda)(t_2-\eta_2))}{\lambda(t_1-\eta_1) - (1-\lambda)(t_2-\eta_2)} = \zeta$$

implying $\hat{\zeta}^{-1} \xrightarrow{P} \zeta^{-1}$.

Moreover, for each i=1,2, direct calculations give

$E(n_i(\hat{\eta}_i-\eta_i))^2 \longrightarrow 2\zeta^{-2}$ and

$E(n_i(\hat{\eta}_i-\eta_i)) \to \zeta^{-1}$ as $n_i \to \infty$

while, after showing uniform integrability of $n_i(\hat{\hat{\eta}}_i-\eta_i)$ and since we already know it converges in distribution to $U_i - \zeta^{-1}$, one gets

$E\left(n_i(\hat{\hat{\eta}}_i-\eta_i)\right)^2 \longrightarrow E(U-\zeta^{-1})^2 = \zeta^{-2}$ and

$E\left(n_i(\hat{\hat{\eta}}_i-\eta_i)\right) \longrightarrow E\left((U-\zeta)^{-1}\right) = 0.$ Hence, $\hat{\hat{\eta}}_i$ achieves asymptotically 50% MSE reduction and 100% bias reduction over $\hat{\eta}_i$.

Also, using Lemmas 3.2.2 one gets that

$$\sqrt{n}\,(\hat{\zeta}^{-1} - \zeta^{-1}) \xrightarrow{L} N(0,\zeta^{-3}\{\lambda(t_1-\eta_1) + (1-\lambda)(t_2-\eta_2)\}^{-1}) \qquad (3.3.22)$$

We shall next show that $[\sqrt{n}\,(\hat{\zeta}^{-1} - \zeta^{-1})]^2$ is uniformly integrable in $n \geq 1$. (3.3.23)

In order to prove (3.3.23) first write

$$[\sqrt{n}\,(\hat{\zeta}^{-1} - \zeta^{-1})]^4 = n^2[(\frac{\sum_{i=1}^{2} n_i(t_i-\hat{\eta}_i)}{R} - \zeta^{-1})I_{[R>1]} + \zeta^{-1}(I_{[R>1]}-1)]^4$$

$$= n^2(\Sigma_{i=1}^2 \frac{n_i(\eta_i-\hat{\eta}_i)}{R} I_{[R>1]} + (\frac{\Sigma_{i=1}^2 n_i(t_i-\eta)}{R} - \zeta^{-1})I_{[R>1]}$$

$$+ \zeta^{-1}I_{[R>1]}-1)^4$$

$$< 3^3n^2[\frac{(\Sigma_{i=1}^2 n_i(\hat{\eta}_i-\eta_i))^4}{R^4} I_{[R>1]} + (\frac{\Sigma_{i=1}^2 n_i(t_i-\eta_i)}{R} - \zeta^{-1})^4 I_{[R>1]}$$

$$+ \zeta^{-4}(I_{[R>1]}-1)^4]$$

Next, using the Schwarz inequality

$$E(\frac{\Sigma_{i=1}^2 n_i(\hat{\eta}_i-\eta_i)}{R} I_{[R>1]})^4$$

$$< (ER^{-8}I_{[R>1]})^{1/2}\,[E(\Sigma_{i=1}^2 n_i(\hat{\eta}_i-\eta_i))^8]^{1/2}$$

$$\leq (ER^{-8} I_{[R>1]})^{1/2} (2^7 E\{(n_1(\hat{\eta}_1-\eta_1))^8 + (n_2(\hat{\eta}_2-\eta_2))^8\})^{1/2}$$

Using arguments similar to (3.2.13) and (3.2.14) one gets $ER^{-8} I_{[R>1]} = O(n^{-8})$. Also, using (3.3.18), simple calculations show that $E(n_i(\hat{\eta}_i-\eta_i))^8 = O(1)$.

Hence

$$n^2 E\left(\frac{\Sigma_{i=1}^2 n_i(\hat{\eta}_i-\eta_i)}{R} I_{[R>1]}\right)^4 = O(n^2 n^{-4} + n^2 n^{-4}) = O(n^{-2}). \quad (3.3.24)$$

Moreover,

$$E\left(\frac{\Sigma_{i=1}^2 n_i(t_i-\eta_i)}{R} - \zeta^{-1}\right)^4 I_{[R>1]}$$

$$= \zeta^{-4} E\left(\frac{1}{R^4}\left[R - \Sigma_{i=1}^2 n_i \zeta(t_i-\eta_i)\right]^4\right) I_{[R>1]}$$

$$\leq \zeta^{-4} [E\, R^{-8} I_{[R>1]}]^{1/2} \cdot [E[R-\Sigma_{i=1}^2 n_i \zeta(t_i-\eta_i)]^8]^{1/2}$$

Again $ER^{-8} I_{[R>1]} = O(n^{-8})$ and

$$E[R - \Sigma_{i=1}^2 n_1 \zeta(t_i-\eta_i)]^8 = E[R_1 - n_1\zeta(t_1-\eta_1) + R_2 - n_2\zeta(t_2-\eta_2)]^8$$

$$\leq E[|R_1-n_1\zeta(t_1-\eta_1)| + |R_2-n_2\zeta(t_2-\eta_2)|]^8$$

$$\leq 2^7 (E|R_1-n_1\zeta(t_1-\eta_1)|^8 + E|R_2-n_2\zeta(t_2-\eta_2)|^8)$$

$$< 2^7(K_1 n_1^4 + K_2 n_2^4)$$

so that

$$n^2 E\left(\frac{\Sigma_{i=1}^2 n_i(t_i-\eta_i)}{R} - \zeta^{-1}\right)^4 I_{[R>1]} < 0\left(n^2 n^{-4}\cdot(n_1^2+n_2^2)\right) = 0(1) \quad (3.3.25)$$

Finally

$$n^2 E\zeta^{-4}(I_{[R>1]}-1)^4 = n^2\zeta^{-4}P[R = 0]$$

$$= n^2\zeta^{-4} e^{-n_1\zeta(t_1-\eta_1) - n_2\zeta(t_2-\eta_2)}. \quad (3.3.26)$$

Hence, combining (3.3.24) through (3.3.26), one gets that

$$\sup_{n>1} E[n^2(\hat{\zeta}^{-1}-\zeta^{-1})^4] < \infty. \quad (3.3.27)$$

Since, now (3.3.23) is an immediate consequence of (3.3.27) it follows that

$$E\sqrt{n}\,(\hat{\zeta}^{-1} - \zeta^{-1}) \to 0 \quad \text{and}$$

$$E[\sqrt{n}\,(\hat{\zeta}^{-1} - \zeta^{-1})]^2 \to \zeta^{-3}\{\lambda(t_1-\eta_1) + (1-\lambda)(t_2-\eta_2)\}^{-1} \text{ as } n \to \infty.$$

Also if at least one η_i $(i=1,2)$ is known the same results can be obtained after minor modificatoins to the proofs given here.

Finally, we consider the case when $\eta_1 = \eta_2 = \eta$ and $\zeta_1 = \zeta_2 = \zeta$. In this case, the MLEs of η and ζ are given respectively by $\hat{\eta} = ZI_{[\eta<Z<t_1]} + t_1 I_{[Z>t_1]}$ and $\hat{\zeta} = \{R/\Sigma_{i=1}^2 n_i(t_i-\hat{\eta})\}I_{[R>1]}$. We can show that $n(\hat{\eta}-\eta)$ converges in distribution to an exponential rv with failure rate ζ while

$\sqrt{n}(\hat{\zeta}^{-1}-\zeta^{-1}) \xrightarrow{L} N(0,\zeta^{-3}\{\lambda(t_1-\eta) + (1-\lambda)(t_2-\eta)\}^{-1})$. Also, by proving the necessary uniform integrability results, one can show that $E[n(\hat{\eta}-\eta)] \to \zeta^{-1}, E[n(\hat{\eta}-\eta)]^2 \to 2\zeta^{-2}$ and $E[n(\hat{\zeta}^{-1}-\zeta^{-1})^2] \to \zeta^{-3}\{\lambda(t_1-\eta) + (1-\lambda)(t_2-\eta)^{-1}\}$. Also defining $\hat{\hat{\eta}} = \hat{\eta} - (n\hat{\zeta})^{-1}I_{[\eta<Z<t_1]}$, it follows that $E[n(\hat{\hat{\eta}}-\eta)]^2 \to \zeta^{-2}$ and $E[n(\hat{\hat{\eta}}-\eta)] \to 0$. Thus, $\hat{\hat{\eta}}$ achieves asymptotically 50% MSE reduction and 100% bias reduction than $\hat{\eta}$. Also if η or ζ is known, all results relating to the parameters of interest still hold. We omit all proofs because of their similarity to the ones given earlier.

CHAPTER FOUR

GENERALIZED LIKELIHOOD RATIO TESTS FOR THE WITH REPLACEMENT CASE

4.1 Introduction

Suppose there are k independent location and scale parameter exponentials and we are interested in testing different hypotheses regarding the equality of the location and/or the equality of the scale parameters when sampling is done with replacement within each group. To be specific, suppose the experiment consists of putting $n_1, n_2, \ldots, n_k$ items to test independently, as explained in Section 1.1. Then, the likelihood function of all observations is given by (1.1.9) namely,

$$L(\underset{\sim}{\eta},\underset{\sim}{\zeta}) = \prod_{i\varepsilon S}\{(n_i\zeta_i)^{r_i}\exp[-n_i\zeta_i(t_i-\eta_i)]I_{[\eta_i<x_{(i1)}<\cdots<x_{(ir_i)}<t_i]}\}$$

$$\times \prod_{i\varepsilon \bar{S}}\{\exp[-n_j\zeta_j(t_j-\eta_j)]\}.$$

The following testing problems are considered.

(i) H_{01}: $\zeta_1 = \cdots = \zeta_k$ against H_{A1} not all ζ_i's are equal, when $\eta_1,\ldots,\eta_k$ are known;

(ii) H_{02}: $\zeta_1 = \cdots = \zeta_k$ against H_{A2}: not all ζ_i's are equal, when $\eta_1 = \eta_2 = \cdots = \eta_k = \eta$ (say), but η is unknown;

(iii) H_{03}: $\zeta_1 = \cdots = \zeta_k$ against H_{A3}: not all ζ_i's are equal;

(iv) H_{04}: $\eta_1 = \cdots = \eta_k$ against H_{A4}: not all η_i's are equal, when $\zeta_1,\ldots,\zeta_k$ are known;

(v) H_{05}: $\eta_1 = \cdots = \eta_k$ against H_{A5}: not all η_i's are equal,

when $\zeta_1 = \cdots = \zeta_k = \zeta$ (say), but ζ is unknown;

(vi) H_{06}: $\eta_1 = \cdots = \eta_k$ against H_{A6}: not all η_i's are equal;

(vii) H_{07}: $\eta_1 = \cdots = \eta_k$ and $\zeta_1 = \cdots = \zeta_k$ against H_{A7}: not all η_i's and/or not all ζ_i's are equal.

The testing problem (i), (ii), and (iii) are considered in Section 4.2. The generalized likelihood ratio test (GLRT) criterion λ is computed, and the asymptotic distribution of $-2\log\lambda$ is given for both the null and local alternatives. In Section 4.3, the testing problems (iv), (v) and (vi) are considered. In this section, the GLRT criterion λ is computed, and the null distribution for $-2\log\lambda$ is derived. The testing problem (vii) is considered in Section 4.4.

Explicit computation of even the asymptotic null distribution of $-2\log\lambda$ becomes quite formidable in this case, but some conservative test procedure is recommended.

4.2 Testing The Equality of Failure Rates

We shall not make a notational distinction between the rv λ or its value. Before carrying out the actual tests, certain preliminary facts are needed. Note that the likelihood ratio is defined on 2^k distinct regions according to all possible (k-tuple) combinations of $\underset{\sim}{r} = (r_1,\ldots,r_k)$ depending on whether r_i is greater than or equal to zero. Let

$$A_j = \{\underset{\sim}{r}: j \text{ of the } r_i\text{'s equal zero}\},\ j=0,1,\ldots,k. \qquad (4.2.1)$$

Hence, for each j, A_j contains $\binom{k}{j}$ elements. Write A_j

$= \bigcup_{\ell=1}^{\binom{k}{j}} B_{j\ell}$, where $(B_{j1},\ldots,B_{j\binom{k}{j}})$ constitutes a partition by elements of A_j. Then

$$-2\log\lambda = \Sigma_{j=0}^{k} \Sigma_{\ell=1}^{\binom{k}{j}} (-2\log\lambda) I_{[\underset{\sim}{r}\varepsilon B_{j\ell}]}. \tag{4.2.2}$$

Note that for $j > 1$,

$$\begin{aligned} &P[(-2\log\lambda)I_{[\underset{\sim}{R}\varepsilon B_{j\ell}]} \neq 0] \\ &< P(\text{at least one } R_i{=}0) < \Sigma_{i=1}^{k} P(R_i{=}0) = \Sigma_{i=1}^{k} \exp[-n_i \zeta_i (t_i - \eta)] \\ &\qquad \to 0 \text{ as } \min_{1<i<k} n_i \to \infty. \end{aligned} \tag{4.2.3}$$

Also, $P(\underset{\sim}{R} \varepsilon B_{01}) \to 1$ as $\min_{1<i<k} n_i \to \infty$. Hence,

$$-2\log\lambda = (-2\log\lambda) I_{[\underset{\sim}{R}\varepsilon B_{01}]} + o_p(1), \tag{4.2.4}$$

where by $o_p(1)$, we mean a random variable which converges in probability to zero as $\min(n_1,\ldots,n_k) \to \infty$.

Next we address the problem of testing H_{01}. From (1.1.9) it follows that for $\underset{\sim}{R} \varepsilon B_{01}$, MLE of ζ_i is $\hat{\zeta}_i = R_i/C_i$ where $C_i = n_i(t_i - \eta_i)$, $i = 1,\ldots,k$. Also, under H_0, MLE of the common failure rate ζ is $\hat{\zeta} = R/C$, where $R = \Sigma_{i=1}^{k} R_i$ and $C = \Sigma_{i=1}^{k} C_i$. Now, for $\underset{\sim}{R} \varepsilon B_{01}$, the GLR test criterion λ is given by

$$\lambda = (R^R / \prod_{i=1}^{k} R_i^{R_i})(\prod_{i=1}^{k} C_i^{R_i}/C^R). \tag{4.2.5}$$

Note that from the central limit theorem,

$$(R_i - C_i\zeta_i)/(C_i\zeta_i)^{1/2} \xrightarrow{L} N(0,1) \text{ as } n_i \to \infty \tag{4.2.6}$$

which implies that $(R_i - C_i\zeta_i)/(C_i\zeta_i)^{1/2} = 0_p(1)$ and

$R_i/(C_i\zeta_i) \xrightarrow{P} 1$ as $n_i \to \infty$.

In order to find the limiting distribution of $-2\log\lambda$, we make the following assumption.

$$\lim_{n\to\infty} n_i/n = \lambda_i,\ 0<\lambda_i<1 \text{ and } \Sigma_{i=1}^{k} \lambda_i = 1, \text{where } n = \Sigma_{i=1}^{k} n_i. \quad (4.2.7)$$

We now prove the first theorem of this section which provides the asymptotic null distribution of $(-2\log\lambda)I_{[\underset{\sim}{R} \in B_{01}]}$. In view of (4.2.4), $-2\log\lambda$ has the same limiting distribution as $(-2\log\lambda)I_{[\underset{\sim}{R} \in B_{01}]}$.

<u>Theorem 4.2.1</u> <u>If</u> (4.2.7) <u>holds</u>, <u>then as</u> $n \to \infty$, <u>under</u> H_{01}: $\zeta_1=\cdots=\zeta_k=\zeta$, $-2\log\lambda \xrightarrow{L} \chi^2_{k-1}$.

<u>Proof</u> Since $\zeta_1=\cdots=\zeta_k=\zeta$, using a Taylor expansion, for $\underset{\sim}{R} \in B_{01}$, one gets from (4.2.5),

$$\begin{aligned}
-2\log\lambda &= 2[\Sigma_{i=1}^{k} R_i\log(R_i/R) - \Sigma_{i=1}^{k} R_i\log(C_i\zeta/C\zeta)] \\
&= 2[\Sigma_{i=1}^{k} R_i\log(R_i/(C_i\zeta)) - R\log(R/(C\zeta))] \\
&= 2[\Sigma_{i=1}^{k} (R_i - C_i\zeta + C_i\zeta)\log(1+(R_i-C_i\zeta)(C_i\zeta)^{-1}) \\
&\quad - (R-C\zeta + C\zeta)\log(1 + (R-C\zeta)(C\zeta)^{-1})] \\
&= 2[\Sigma_{i=1}^{k}(R_i-C_i\zeta + C_i\zeta) \times
\end{aligned}$$

$$\Big\{\frac{R_i-C_i\zeta}{C_i\zeta} - \frac{(R_i-C_i\zeta)^2}{2C_i^2\ \zeta^2} + \frac{(R_i-C_i\zeta)^3}{3!C_i^3\zeta^3\left(1+\phi_i\frac{R_i-C_i\zeta}{C_i\zeta}\right)^3}\Big\}$$

$$- (R-C\zeta+C\zeta)\Big\{\frac{R-C\zeta}{C\zeta} - \frac{(R-C\zeta)^2}{2C^2\zeta^2} + \frac{(R-C\zeta)^3}{3!(C\zeta)^3\left(1+\phi\ \frac{R-C\zeta}{C\zeta}\right)^3}\Big\} \quad (4.2.8)$$

where $0<\phi_i<1 (i=1,2,\ldots,k)$ and $0<\phi<1$.

Then, after multiplying term by term in (4.2.8) and simplifying one gets that

$$-2\log\lambda I_{[\underset{\sim}{R}\varepsilon B_{01}]} = \sum_{j=1}^{4} \left((R_j - C_j\zeta)/(C_j\zeta)^{1/2}\right)^2 - \left((R-C\zeta)/(C\zeta)^{1/2}\right)^2$$

$$- \sum_{j=1}^{k} \left((R_j - C_j\zeta)/(C_j\zeta)^{1/2}\right)^3 (C_j\zeta)^{-1/2}$$

$$+ \sum_{j=1}^{k} \frac{\left((R_j - C_j\zeta)/(C_j\zeta)^{1/2}\right)^3 (C_j\zeta)^{-1/2}}{3\left(1 + \phi_j\left(\frac{R_j - C_j\zeta}{(C_j\zeta)^{1/2}}\right)(C_j\zeta)^{-1/2}\right)^3}$$

$$+ \sum_{j=1}^{k} \frac{\left((R_j - C_j\zeta)/(C_j\zeta)^{1/2}\right)^4 (C_j\zeta)^{-1}}{3\left(1 + \phi_j\left(\frac{R_j - C_j\zeta}{(C_j\zeta)^{1/2}}\right)(C_j\zeta)^{-1/2}\right)^3}$$

$$+ \left((R-C\zeta)/\sqrt{C\zeta}\right)^3 (C\zeta)^{-1/2}$$

$$- \frac{\left((R-C\zeta)/(C\zeta)^{1/2}\right)^3 (C\zeta)^{-1/2}}{3\left(1 + \phi\left(\frac{R-C\zeta}{(C\zeta)^{1/2}}\right)(C\zeta)^{-1/2}\right)^3}$$

$$- \frac{\left((R-C\zeta)/(C\zeta)^{1/2}\right)^4 (C\zeta)^{-1}}{3\left(1 + \phi\left(\frac{R-C\zeta}{(C\zeta)^{1/2}}\right)(C\zeta)^{-1/2}\right)^3}$$

$$\times I_{[\underset{\sim}{R}\varepsilon B_{01}]}. \qquad (4.2.9)$$

Next, using (4.2.6) and (4.2.4) and the independence of groups, one gets that

$$\frac{R-C\zeta}{\sqrt{C\zeta}} = \sum_{i=1}^{k} \frac{R_j - C_j\zeta}{(C_j\zeta)^{1/2}} \cdot (C_j/C)^{1/2}$$

$$\xrightarrow{L} \sum_{i=1}^{k} \left(\frac{\lambda_i(t_i-\eta_i)}{\sum_{j=1}^{k} \lambda_j(t_j-\eta_j)} \right)^{1/2} Z \tag{4.2.10}$$

where Z is distributed as a normal variate with mean zero and variance one. Hence

$$(R-C\zeta)(C\zeta)^{-1/2} = 0_p(1). \tag{4.2.11}$$

Next, using (4.2.3), (4.2.6), and (4.2.11), it follows from (4.2.9) that

$$-2\log\lambda I_{[\underset{\sim}{R}\varepsilon B_{01}]}$$

$$= \{ \sum_{i=1}^{k} (R_i-C_i\zeta)^2(C_i\zeta)^{-1} - (R-C\zeta)^2(C\zeta)^{-1}\} I_{[\underset{\sim}{R}\varepsilon B_{01}]} + o_p(1) \tag{4.2.12}$$

Hence, for proving the theorem, it suffices to show that

$$Q = \{\Sigma_{i=1}^{k}(R_i-C_i\zeta)^2(C_i\zeta)^{-1} - (R-C\zeta)^2(C\zeta)^{-1}\} I_{[\underset{\sim}{R}\varepsilon B_{01}]} \xrightarrow{L} \chi^2_{k-1}$$

under H_{01}. (4.2.13)

To prove (4.2.13), write $Q = (\underset{\sim}{Y}'\underset{\sim}{A}\underset{\sim}{Y})\, I_{[\underset{\sim}{R}\ \varepsilon\ B_{01}]}$, where $\underset{\sim}{Y} = (Y_1,\ldots,Y_k)'$ with $Y_j = (R_j-C_j\zeta)/(C_j\zeta)^{1/2}$ $(j=1,\ldots,k)$ and $A_1 = \underset{\sim}{I}_k - \underset{\sim}{u}_1\underset{\sim}{u}_1'$, $\underset{\sim}{u}_1' = ((C_1/C)^{1/2},\ldots,(C_k/C)^{1/2})$.

Note that as $n \to \infty$, in view of (4.2.7), $\underset{\sim}{A}_1 \to \underset{\sim}{I}_k - \underset{\sim}{d}_1\underset{\sim}{d}_1'$, where $d_1' = ((\lambda_1^{1/2}(t_1-\eta_1)^{1/2},\ldots,\lambda_k^{1/2}(t_k-\eta_k)^{1/2})[\Sigma_{i=1}^{k}\lambda_i(t_i-\eta_i)]^{-1/2}$.

Also, using the multivariate central limit theorem, under H_{01}, $\underset{\sim}{Y} \xrightarrow{L} N_k(\underset{\sim}{0},\underset{\sim}{I}_k)$.

In what follows we make use of the following lemma:

<u>Lemma 4.2.1</u> Let $\underset{\sim}{X} = (X_1,\ldots,X_k)'$ be $N_k(\underset{\sim}{u},I_k)$, I_k the identity matrix, and let $C_{k\times k}$ be a symmetric matrix. Then the quadratic

form X'CX has a (possibly noncentral) chi-squared distribution if and only if C is idempotent, that is $C^2 = C$, in which case the degrees of freedom is rank (C) = trace (C) and the noncentrality parameter is $\underset{\sim}{u}'C\underset{\sim}{u}'$.

Now, $\underset{\sim}{I}_k - \underset{\sim}{d}_1\underset{\sim}{d}_1'$ is symmetric, idempotent with rank $(\underset{\sim}{I}_k - \underset{\sim}{d}_1\underset{\sim}{d}_1')$ $= tr(\underset{\sim}{I}_k - \underset{\sim}{d}_1\underset{\sim}{d}_1') = k-1$. Hence,using the lemma,

$Y'(\underset{\sim}{I}_k - \underset{\sim}{d}_1\underset{\sim}{d}_1')Y \xrightarrow{L} \chi^2_{k-1}$ and

since $Y'(\underset{\sim}{A}_1 - [I_k - \underset{\sim}{d}_1\underset{\sim}{d}_1'])Y \xrightarrow{P} 0$ as $n \to \infty$ it follows using Slutsky's that $Y'A_1Y \xrightarrow{L} \chi^2_{k-1}$. Since $I_{[\underset{\sim}{R} \in B_{01}]} \xrightarrow{P} 1$ as $n \to \infty$,

one gets (4.2.13). □

Next consider the sequence of local alternatives $\zeta_i = \zeta + \Delta_i n_i^{-1/2}$ $(i=1,2,\ldots,k)$. We use the Taylor expansion for $-2\log\lambda$ as in (4.2.8). First write $(R_i-C_i\zeta)/(C_i\zeta)^{1/2} = (R_i-C_i\zeta_i)/(C_i\zeta)^{1/2} + C_i(\zeta_i-\zeta)(C_i\zeta)^{-1/2} = (\zeta_i/\zeta)^{1/2}(R_i-C_i\zeta_i)(C_i\zeta_i)^{-1/2} + \Delta_i C_i^{1/2} n_i^{-1/2}\zeta^{-1/2}$. Since $C_i = n_i(t_i - \eta_i)$, using the multivariate central limit theorem, it follows that $\underset{\sim}{Y} = ((R_1-C_1\zeta)(C_1\zeta)^{-1/2}, \ldots,(R_k-C_k\zeta)(C_k\zeta)^{-1/2}) \xrightarrow{L} N(\underset{\sim}{\delta},\underset{\sim}{I}_k)$, where $\underset{\sim}{\delta} = (\delta_1,\ldots,\delta_k)'$ with $\delta_i = \Delta_i((t_i-\eta_i)/\zeta)^{1/2}$. Arguing as in the null case, it follows now that $Q \xrightarrow{L} \chi^2_{k-1}(\tau_1)$, where the noncentrality parameter

$$\tau_1 = \underset{\sim}{\delta}'(\underset{\sim}{I}_k-\underset{\sim}{d}\,\underset{\sim}{d}')\underset{\sim}{\delta} = \zeta^{-1}[\Sigma_{i=1}^k\Delta_i^2(t_i-\eta_i)$$

$$- \{\sum_{i=1}^k \Delta_i(t_i-\eta_i)^{1/2}(\lambda_i(t_i-\eta_i)\Big|\sum_{i=1}^k\lambda_i(t_i-\eta_i))^{1/2}\}^2.$$

Next we consider testing H_{02}. In this case, the MLE of η is $\hat{\eta} = \min_{1<i<k} X_{(i1)}$, and the MLE of ζ_i is $\hat{\zeta}_i = R_i/(n_i(t_i-\hat{\eta}))$ for $\underset{\sim}{R} \in B_{01}$. Let

$\hat{C}_i = n_i(t_i-\hat{\eta})$ and $\hat{C} = \Sigma_{i=1}^{k}\hat{C}_i$. Under H_{02}, the MLE of the common failure rate ζ is $\hat{\zeta} = R/\hat{C}$. Hence, from (1.1.9) for $\underset{\sim}{R} \in B_{01}$,

$$\lambda = \prod_{i=1}^{k} (\hat{C}_i/R_i)^{R_i}(R/\hat{C})^R. \tag{4.2.14}$$

As in (4.2.8), for $\underset{\sim}{R} \in B_{01}$,

$-2\log\lambda$

$$= 2[\Sigma_{i=1}^{k}(R_i-\hat{C}_i\zeta+\hat{C}_i\zeta)\{\frac{R_i-\hat{C}_i\zeta}{\hat{C}_i\zeta} - \frac{(R_i-\hat{C}_i\zeta)^2}{2\hat{C}_i^{\,2}\zeta^2} + \frac{(R_i-\hat{C}_i\zeta)^3}{3\hat{C}_i^{\,3}\zeta^3(1+\phi_i\frac{R_i-\hat{C}_i\zeta}{\hat{C}_i\zeta})^3}$$

$$-(R-\hat{C}\zeta+\hat{C}\zeta)\{\frac{R-\hat{C}\zeta}{\hat{C}\zeta} - \frac{(R-\hat{C}\zeta)^2}{2\hat{C}^2\zeta^2} + \frac{(R-\hat{C}\zeta)^3}{3\hat{C}^3\zeta^3(1+\phi\frac{R-\hat{C}\zeta}{\hat{C}\zeta})^3}\} \tag{4.2.15}$$

Next, write

$(R_i-\hat{C}_i\zeta)(\hat{C}_i\zeta)^{-1}$

$$= (C_i/\hat{C}_i)(R_i-C_i\zeta)(C_i\zeta)^{-1/2} + n_i^{1/2}(\hat{\eta}-\eta)(t_i-\hat{\zeta})^{-1/2}\zeta^{-1/2} \tag{4.2.16}$$

(where $C_i = n_i(t_i-\eta)$ $i=1,2,\ldots k$ and η is unknown)

Note that $\hat{\eta} \xrightarrow{P} \eta$ as $n \to \infty$, and $n(\hat{\eta}-\eta)$ converges to an exponential random variable as $n \to \infty$. Thus, in view of (4.2.7) $C_i/\hat{C}_i \xrightarrow{P} 1$ as $n \to \infty$ and $n_i^{1/2}(\hat{\eta}-\eta) \xrightarrow{P} 0$ as $n \to \infty$. Hence, writing $Z_i = (R_i - \hat{C}_i\zeta)(\hat{C}_i\zeta)^{1/2}(i=1,\ldots,k)$, and defining $W_i = (R_i - C_i\zeta)(C_i\zeta)^{-1/2}$ for $(i=1,\ldots k)$, it follows from (4.2.16) and the previous remarks that $Z_i - W_i \xrightarrow{P} 0$. Note that the Z_i's are not independent since they all share the same $\hat{\eta}$. However

$W_i \xrightarrow{L} N(0,1)$ for $i=1,2,\dots k$ and the W_i's are independently distributed. It follows using the multivariate central limit theorem that $\underset{\sim}{W} = (W_1,\dots W_k)' \xrightarrow{L} N_k(\underset{\sim}{0},\underset{\sim}{I}_k)$ under H_{02}.

Using Slutskys theorem and the Cramer-Wold device, one gets

$\underset{\sim}{Z} = (Z_1,\dots,Z_k)' \xrightarrow{L} N(0,I_k)$ under H_{02}.

Now from (4.2.15)

$$(-2\log\lambda)I_{[\underset{\sim}{R}\varepsilon B_{01}]} = (Z'\hat{A}_2 Z)I_{[\underset{\sim}{R}\varepsilon B_{01}]} + o_p(1) \qquad (4.2.17)$$

where $\hat{A}_2 = I_k - \hat{u}_2\hat{u}_2'$ and $\hat{u}_2' = \left((\hat{C}_1/\hat{C})^{1/2},\dots,(\hat{C}_k/\hat{C})^{1/2}\right)$. Note that $Z'(A_2-\hat{A}_2)Z \xrightarrow{P} 0$, where $A_2 = I_k - \underset{\sim}{d}_2\underset{\sim}{d}_2'$ and

$d_2' = \left(\lambda_1(t_1-\eta)^{1/2},\dots,\lambda_k(t_k-\eta)^{1/2}\right)\left(\sum_{i=1}^{k}\lambda_i(t_i-\eta)\right)^{-1/2}$ and η is unknown.

Arguing as in the case of H_{01}, it follows that

$(-2\log\lambda)\ I_{[\underset{\sim}{R}\varepsilon B_{01}]} \xrightarrow{L} \chi^2_{k-1}$ under H_{02}.

Once again, consider the sequence of local alternatives

$\zeta_i = \zeta + \Delta_i n_i^{-1/2}$, $i=1,2,\dots k$. Using the first line of (4.2.16), one gets

$$Z_i = \left(\frac{\zeta_i}{\zeta}\right)^{1/2}(\hat{C}_i\zeta_i)^{-1/2}(R_i-\hat{C}_i\zeta_i) + \hat{C}_i^{1/2}\zeta^{-1/2}(\zeta_i-\zeta)$$

$$= (\zeta_i/\zeta)^{1/2}(\hat{C}_i\zeta_i)^{-1/2}(R_i-\hat{C}_i\zeta_i) + \hat{C}_i^{1/2}\zeta^{-1/2}\Delta_i n_i^{-1/2}$$

$$\xrightarrow{L} N(\delta_i^*,\ 1)$$

where $\delta_i^* = \Delta_i(t_i-\eta)^{1/2}/\zeta^{1/2}$, for $i=1,\dots,k$.

Since the Z_i's are not independent, we write

$W_i = (\zeta_i/\zeta)^{1/2}(C_i\zeta_i)^{-1/2}(R_i-C_i\zeta_i) + C_i^{1/2}\zeta^{-1/2}(\zeta_i-\zeta)$ $i=1,\ldots k$ and note that $W_i \xrightarrow{L} N(\delta_i^*,1)$ and $W_i-Z_i \xrightarrow{P} 0$. Also, the W_i's are independent and hence via the multivariate central limit theorem $\underset{\sim}{W} = (W_1,\ldots,W_k)' \xrightarrow{L} N(\underset{\sim}{\delta}^*,\underset{\sim}{I}_k)$ where $\underset{\sim}{\delta}^* = (\delta_1^*,\ldots\delta_k^*)'$. Arguing as before for the given sequence of local alternatives,

$-2\log\lambda \to \chi^2_{k-1}(\tau_2)$ where $\tau_2 =$

$$\zeta^{-1}[\Sigma_{i=1}^k \Delta_i^2(t_i-\eta)-\{\Sigma_{i=1}^k \Delta_i(t_i-\eta)^{1/2}(\lambda_i(t_i-\eta)/\Sigma_{i=1}^k\lambda_i(t_i-\eta))^{1/2}\}^2$$

Finally, in this section, we consider testing H_{03}. In this case, the MLE of η_i is $\hat{\eta}_i = X_{(i1)}$ and for $\underset{\sim}{R} \in B_{01}$, ζ_i has MLE $\hat{\zeta}_i = R_i/\hat{C}_{i0}$, where $\hat{C}_{i0} = n_i(t_i-\hat{\eta}_i)$, $i = 1,\ldots,k$. Under $H_{03}:\zeta_1 =\ldots=\zeta_k = \zeta$, the MLE of the common failure rate ζ is $\hat{\zeta} = R/\hat{C}_0$, where $\hat{C}_0 = \Sigma_{i=1}^k \hat{C}_{i0}$. Hence, for $\underset{\sim}{R} \in B_{01}$, the GLRT criterion λ equals

$$\lambda = \{\prod_{i=1}^k (\hat{C}_{i0}/R_i)^{R_i}\}(R/\hat{C}_0)^R. \qquad (4.2.18)$$

Hence, for $\underset{\sim}{R} \in B_{01}$,

$$-2\log\lambda = 2[\Sigma_{i=1}^k R_i\log(R_i/R) - \Sigma_{i=1}^k R_i\log(\hat{C}_{i0}/\hat{C}_0\zeta)]$$

$$= 2[\Sigma_{i=1}^k (R_i-\hat{C}_{i0}\zeta + \hat{C}_{i0}\zeta)\log(1+ \frac{R_i-\hat{C}_{i0}\zeta}{\hat{C}_{i0}\zeta})$$

$$- (R-\hat{C}_0\zeta + \hat{C}_0\zeta)\log\left(1 + \frac{R-\hat{C}_0\zeta}{\hat{C}_0\zeta}\right)]. \qquad (4.2.19)$$

Note that since

$$(R_i-\hat{C}_{i0}\zeta)(\hat{C}_{i0}\zeta)^{-1/2} = (C_{i0}/\hat{C}_{i0})^{1/2}[(R_i-C_{i0}\zeta)(C_{i0}\zeta)^{-1/2}]$$

$$+ \{(\hat{\eta}_i-\eta_i)^{1/2}/(t_i-\hat{\eta}_i)^{1/2}\}\, n_i^{1/2}\zeta^{1/2}, \quad (4.2.20)$$

where $C_{i0} = n_i(t_i-\eta_i)$ $(i=1,\ldots,k)$, $\hat{\eta}_i \xrightarrow{P} \eta_i$ and n_i $(\hat{\eta}_i-\eta_i) \xrightarrow{L}$ an exponential random variable, using the multivariate central limit theorem, it follows from (4.2.20) that under H_0,

$$((R_1-\hat{C}_{10}\zeta)(\hat{C}_{10}\zeta)^{-1/2},\ldots,(R_k-\hat{C}_{k0}\zeta)^{1/2}(\hat{C}_{k0}\zeta)^{-1/2})' \xrightarrow{L} N_k(\underset{\sim}{0},\underset{\sim}{I}_k), \qquad (4.2.21)$$

as $n \to \infty$. Hence, $(R_i-\hat{C}_{i0}\zeta)(\hat{C}_{i0}\zeta)^{-1/2} = 0_p(1)$ and $R_i/(\hat{C}_{i0}\zeta) \xrightarrow{P} 1$ as $n \to \infty$. It is now easy to see from (4.2.19) that

$$(-2\log\lambda)I_{[\underset{\sim}{R} \in B_{01}]} = (\underset{\sim}{U}'\hat{\underset{\sim}{A}}_3\underset{\sim}{U})\, I_{[\underset{\sim}{R} \in B_{01}]} + o_p(1), \qquad (4.2.22)$$

where $\underset{\sim}{U} = (U_1,\ldots,U_k)'$with $U_i = (R_i - \hat{C}_{i0}\zeta)(\hat{C}_{i0}\zeta)^{-1/2}$, $i = 1,\ldots,k$. $\hat{A}_3 = \underset{\sim}{I}_k-\underset{\sim}{u}_3\underset{\sim}{u}_3'$, $\underset{\sim}{u}_3' = ((\hat{C}_{10}/\hat{C}_0)^{1/2},\ldots,(\hat{C}_{k0}/\hat{C}_0)^{1/2})$ and $\hat{C}_0 = \Sigma_{i=1}^k \hat{C}_{i0}$ Arguing as in the case of H_{01}, it follows that under H_{03} $(-2\log\lambda)I_{[\underset{\sim}{R} \in B_{01}]} \xrightarrow{L} \chi^2_{k-1}$. Also, since $\hat{\eta}_i \xrightarrow{P} \eta$ as $n_i \to \infty$, for local alternatives $\zeta_i = \zeta + \Delta_i n_i^{-1/2}$, one gets
$-2\log\lambda \xrightarrow{L} \chi^2_{k-1}(\tau_3)$, where

$$\tau_3 = \zeta^{-1}[\Sigma_{i=1}^k \Delta_i^2(t_i-\eta_i) - \{\Sigma_{i=1}^k \Delta_i(t_i-\eta_i)^{1/2}(\lambda_i(t_i-\eta_i)|\Sigma_{i=1}^k\lambda_i(t_i-\eta_i))^{1/2}\}$$

and the η_i's are all unknown for $i = 1,\ldots,k$.

4.3. Testing The Equality of Locations

First we test H_{04}. Note that $\underset{\sim}{R} \in B_{01}$ <=> $X_{(i1)} < t_i$ for all $i=1,\ldots,k$. The MLE of η_i is $\hat{\eta}_i = X_{(i1)}$, while under H_{04}, the MLE of the common location parameter η is $\hat{\eta} = \min_{1\le i\le k} X_{(i1)}$. Hence, from (1.1.9), for $\underset{\sim}{R} \in B_{01}$, the GLRT criterion λ is given by

$$\lambda = \exp[-\Sigma_{i=1}^k n_i\zeta_i(\hat{\eta}_i - \hat{\eta})]. \qquad (4.3.1)$$

Thus, for $\underset{\sim}{R} \in B_{01}$,

$$-2\log\lambda = 2\Sigma_{i=1}^{k} n_i\zeta_i (\hat{\eta}_i - \hat{\eta})$$
$$= 2[\Sigma_{i=1}^{k} n_i\zeta_i (\hat{\eta}_i - \eta) - \Sigma_{i=1}^{k} n_i\zeta_i (\hat{\eta}_i - \eta)]. \quad (4.3.2)$$

The first theorem of this section finds the asymptotic null distribution of $-2\log\lambda$. Specifically, we prove the following theorem.

Theorem 4.3.1 If (4.2.7) holds, then under H_{04},

$$(-2\log\lambda)I_{[\underset{\sim}{R} \in B_{01}]} \xrightarrow{L} \chi^2_{2k-2}.$$

Proof Suppose $Y_1,\ldots,Y_k$ are independently distributed, Y_i having pdf

$$f(y_i) = n_i\zeta_i \exp[-n_i\zeta_i(y_i-\eta_i)]I_{[y_i>\eta_i]}, \quad (4.3.3)$$

$i = 1,\ldots,k$. Then, it is easy to see that under H_{04},

$$(X_{(11)},\ldots,X_{(k1)}) \overset{d}{=} (W_1,\ldots,W_k), \quad (4.3.4)$$

where W_i's are independently distributed with

$$W_i = Y_i I_{[\eta<Y_i<t_i]} + t_i I_{[Y_i>t_i]}, \; i=1,\ldots,k. \quad (4.3.5)$$

In the above '$\overset{d}{=}$' means equal in distribution.

Next, observe that

$$P(W_i \neq Y_i) = P(Y_i>t_i) = \exp[-n_i\zeta_i(t_i-\eta_i)] \to 0 \text{ as } n_i \to \infty.$$

Hence, $\quad Y_i - W_i \xrightarrow{P} 0$ (as $n_i \to \infty$). $\quad$ (4.3.6)

Thus, from (4.3.2) and using Slutsky's theorem, $(-2\log\lambda)I_{[\underset{\sim}{R} \in B_{01}]}$ has the same limiting distribution as

$$H = 2[\Sigma_{i=1}^{k} n_i\zeta_i(Y_i-\eta) - (\Sigma_{i=1}^{k} n_i\zeta_i)(Y-\eta)],$$ where $Y = \min(Y_1,\ldots,Y_k)$.

Note that, under H_{04}, $\eta_1 = \ldots = \eta_k$, Y is complete sufficient for η and H is ancillary, i.e. H has a distribution which does not

depend on η. Hence, using Basu's theorem, H and Y are independently distributed. Also, $\Sigma_{i=1}^{k} n_i\zeta_i(Y_i-\eta) \sim \chi^2_{2k}$ under H_0, while $2(\Sigma_{i=1}^{k} n_i\zeta_i)(Y-\eta) \sim \chi^2_2$ under H_0. Accordingly, $-2\log\lambda \sim \chi^2_{2k-2}$ under H_0. □

Remark 1 In the uncensored case, $-2\log\lambda$ is distributed exactly as χ^2_{2k-2} (see for example Hogg (1956)).

Next we consider testing H_{05}. In this case, the MLE of η_i is $\hat{\eta}_i = X_{(i1)}$ and the MLE of the common scale parameter ζ is $\hat{\zeta} = R/\Sigma_{i=1}^{k} n_i(t_i-\hat{\eta}_i)$. Under H_{05}, the MLE of the common location parameter η is $\hat{\eta} = \min_{1\le i\le k} X_{(i1)}$ and the MLE of ζ is $\zeta = R/\Sigma_{i=1}^{k} n_i(t_i-\hat{\eta})$. Then for $\underset{\sim}{R} \in B_{01}$, the GLRT criterion λ is

$$\lambda = (\zeta/\hat{\zeta})^R = \{\Sigma_{i=1}^{k} n_i(t_i-\hat{\eta}_i)/\Sigma_{i=1}^{k} n_i(t_i-\hat{\eta})\}^R. \qquad (4.3.7)$$

Accordingly, for $\underset{\sim}{R} \in B_{01}$,

$$-2\log\lambda = -2R\log\left[1 - \frac{\Sigma_{i=1}^{k} n_i(\hat{\eta}_i-\hat{\eta})}{\Sigma_{i=1}^{k} n_i(t_i-\hat{\eta})}\right]. \qquad (4.3.8)$$

Using the inequality $\log(1-x)\le -x$ for $0<x<1$, it follows from (4.3.8) that

$$(-2\log\lambda)I_{[\underset{\sim}{R} \in B_{01}]}$$

$$\ge \{R/(\Sigma_{i=1}^{k} n_i(t_i-\hat{\eta}))\}\{2\Sigma_{i=1}^{k} n_i(\hat{\eta}_i-\hat{\eta})\}I_{[\underset{\sim}{R}\in B_{01}]}. \qquad (4.3.9)$$

We have proved already in connection with testing H_{04} that $2\Sigma_{i=1}^{k} n_i(\hat{\eta}_i-\hat{\eta}) \xrightarrow{L} \zeta^{-1}\chi^2_{2k-2}$ under H_0. Moreover, under

H_0, $\hat{\eta} \xrightarrow{P} \eta$ and $R/\{\Sigma_{i=1}^k n_i(t_i-\eta)\} \xrightarrow{P} \zeta$. Thus, one gets

$$\text{right hand side of (4.3.9)} \xrightarrow{L} \chi^2_{2k-2}. \tag{4.3.10}$$

Next using the inequality $\log(1-x) > -x - \frac{x^2}{2(1-x)}$ for $0<x<1$, it follows from (4.3.8) that

$$(-2\log\lambda)I_{[\underset{\sim}{R} \,\varepsilon\, B_{01}]}$$

$$<[\{R/((\Sigma_{i=1}^k n_i(t_i-\hat{\eta}))\}\{2\Sigma_{i=1}^k n_i(\hat{\eta}_i-\hat{\eta})\}I_{[\underset{\sim}{R}\varepsilon B_{01}]}$$

$$+ \{R/(\Sigma_{i=1}^k n_i(t_i-\hat{\eta}))\}\{\Sigma_{i=1}^k n_i(t_i-\hat{\eta})\}^{-1}\{\Sigma_{i=1}^k n_i(\hat{\eta}_i-\hat{\eta})\}^2 I_{[\underset{\sim}{R}\varepsilon B_{01}]}]. \tag{4.3.11}$$

We have already seen that the first term in the right hand side of (4.3.11) converges in distribution to χ^2_{2k-2} under H_0. Since $R/\Sigma_{i=1}^k n_i(t_i-\hat{\eta}) \xrightarrow{P} \zeta$, $\{\Sigma_{i=1}^k n_i(\hat{\eta}_i-\hat{\eta})\}^2 I_{[\underset{\sim}{R} \,\varepsilon\, B_{01}]}$ converges in distribution to $[\zeta^{-1}/2\ \chi^2_{2k-2}]^2$ under H_0 and $\hat{\eta} \xrightarrow{P} \eta$ as $n \to \infty$, it follows that the second term in the right hand side of (4.3.11) converges in probability to zero. Thus, from (4.3.9) - (4.3.11) it follows that under (4.2.7),

$$(-2\log\lambda)I_{[\underset{\sim}{R} \,\varepsilon\, B_{01}]} \xrightarrow{L} \chi^2_{2k-2} \text{ as } n \to \infty.$$

Finally, we consider testing H_{06}. For $\underset{\sim}{R} \,\varepsilon\, B_{01}$, note that the MLE of η_i is $\hat{\eta}_i = X_{(i1)}$, while the MLE of ζ_i is $\hat{\zeta}_i = R_i/(n_i(t_i-\hat{\eta}_i))$.

Under H_{06}, the MLE of the common location parameter η is $\hat{\eta}_i = \min_{1\le i\le k} X_{(i1)}$, while the MLE of ζ_i is $\hat{\zeta}_i = R_i/(n_i(t_i-\hat{\eta}))$. Now from (1.1.9), it follows that for $\underset{\sim}{R} \,\varepsilon\, B_{01}$ the GLRT criterion is given by

$$\lambda = \prod_{i=1}^{k}\{(t_i-\hat{\eta}_i)/(t_i-\eta)\}^{R_i}\{(t_i-\hat{\eta})/(t_i-\eta)\}^{-R_i}. \tag{4.3.12}$$

Note that under H_{06}, since $P(\hat{\eta}_i > x \mid R_i = r_i > 0) = \{(t_i- x)/(t_i- \eta)\}^{r_i}$ for $\eta < x_i < t_i$, it follows that conditional on $R_i = r_i(i=1,\dots,k)$, under H_{06}, $V_i = \{(t_i- \hat{\eta}_i)/(t_i-\eta)\}^{R_i}$ are iid uniform (0,1). Also, $P(\hat{\eta}>z \mid \underset{\sim}{R}= r,\ r_i > 0,\ i=1,\dots,k)$ $= \prod_{i=1}^{k}P(\hat{\eta} >z \mid R_i= r_i(>0)) = \prod_{i=1}^{k}\{(t_i-z)/(t_i-\eta)\}^{r_i}$ under H_0. Thus, conditional on $R_i= r_i > 0 \quad 1\leq i\leq k$,
$V = \prod_{i=1}^{k}\{(t_i- \hat{\eta})/(t_i- \eta)\}^{R_i} \sim$ uniform (0,1). Next, observe that conditional on $\underset{\sim}{R} \in B_{01}$, $\hat{\eta}$ is complete sufficient for η, while from (4.3.12) it follows that λ has a distribution which does not depend on η under H_0. Now, under H_{06}, conditional on $\underset{\sim}{R} \in B_{01}$, $\Sigma_{i=1}^{k}(-2\log V_i) \sim \chi^2_{2k}$ and $-2\log V \sim \chi^2_2$. Consequently, conditional on $\underset{\sim}{R} \in B_{01}$, $-2\log\lambda \sim\chi^2_{2k-2}$. Moreover, $I_{[\underset{\sim}{R} \in B_{01}]} \xrightarrow{P} 1$ as $n \to \infty$, and this implies that $(-2\log\lambda) \xrightarrow{L} \chi^2_{2k-2}$ under H_{06}.

4.4 Testing For Location and Scale Parameters

In this section, we test H_{07}. Note that for $\underset{\sim}{R} \in B_{01}$, the MLE of η_i is $\hat{\eta}_i = X_{(i1)}$, while the MLE of ζ_i is $\hat{\zeta}_i = R_i/(n_i(t_i-\hat{\eta}_i))$. Under H_0, for $\underset{\sim}{R} \in B_{01}$ the MLE of the common location parameter η is $\hat{\eta} = \min_{1\leq i\leq k} X_{(i1)}$, while the MLE of the common scale parameter ζ is $\hat{\zeta} = R/(\Sigma_{i=1}^{k} n_i(t_i-\hat{\eta}))$. Hence, for $\underset{\sim}{R} \in B_{01}$, from (1.1.9), the GLRT criterion λ is given by

$$\lambda = \hat{\zeta}^R/(\prod_{i=1}^{k}\hat{\zeta}_i^{R_i}). \tag{4.4.1}$$

Hence, for $\underset{\sim}{R} \in B_{01}$,

$$\begin{aligned}-2\log\lambda &= 2(\Sigma_{i=1}^{k} R_i\log\hat{\zeta}_i - R\log\hat{\zeta}) \\ &= 2[\Sigma_{i=1}^{k} R_i\log(\hat{\zeta}_i/\zeta) - R\log(\hat{\zeta}/\zeta)]. \end{aligned}\tag{4.4.2}$$

Write $\hat{C}_{i0} = n_i(t_i - \hat{\eta}_i)$ and $\hat{C}_i = n_i(t_i - \hat{\eta})$ as in Section 4.2. Hence, for $\underset{\sim}{R} \varepsilon B_{01}$, it follows from (4.4.2) that

$$-2\log\lambda$$
$$= [\Sigma_{i=1}^{k}(R_i - \hat{C}_i\zeta + \hat{C}_i\zeta)\log\left(1+ \frac{R_i-\hat{C}_i\zeta}{\hat{C}_i\zeta}\right)$$
$$-(R- \hat{C}\zeta + \hat{C}\zeta)\log\left(1 + \frac{R-\hat{C}\zeta}{\hat{C}\zeta}\right)]$$
$$-2\Sigma_{i=1}^{k} R_i\log\left(1- \frac{n_i(\hat{\eta}_i - \hat{\eta})}{n_i(t_i-\hat{\eta})}\right), \tag{4.4.3}$$

where $\hat{C} = \Sigma_{i=1}^{k} \hat{C}_i$. Now, combine the arguments used for testing H_{01} as well as H_{05}. This leads to

$$(-2\log\lambda)I_{[\underset{\sim}{R} \varepsilon B_{01}]} = (Q_1 + Q_2)I_{[\underset{\sim}{R} \varepsilon B_{01}]} + o_p(1), \tag{4.4.4}$$

as $n \to \infty$, where

$$Q_1 = \Sigma_{i=1}^{k} (R_i-\hat{C}\zeta)^2(\hat{C}_i\zeta)^{-1} - (R-\hat{C}\zeta)^2(\hat{C}\zeta)^{-1}; \tag{4.4.5}$$

$$Q_2 = \Sigma_{i=1}^{k} \left(R_i/\{n_i(t_i-\hat{\eta})\}\right)\left(2n_i(\hat{\eta}_i - \hat{\eta})\right). \tag{4.4.6}$$

Under H_{07}, conditional on $\underset{\sim}{R} \varepsilon B_{01}$, $Q_1 \xrightarrow{L} \chi^2_{k-1}$. Also, since $R_i/(n_i(t_i-\hat{\eta}_i)) \xrightarrow{P} \zeta$ for all $i=1,\ldots,k$ and $2\Sigma_{i=1}^{k} n_i(\hat{\eta}_i - \hat{\eta}) \xrightarrow{L} \zeta^{-1} \chi^2_{2(k-1)}$ conditional on $\underset{\sim}{R} \varepsilon B_{01}$, it follows that $Q_2 \xrightarrow{L} \chi^2_{2(k-1)}$. Also $I_{[\underset{\sim}{R} \varepsilon B_{01}]} \xrightarrow{P} 1$ as $n \to \infty$. However, Q_1 and Q_2 are <u>not</u> independent. Thus, under H_{07}, $-2\log\lambda \xrightarrow{L} Y_1 + Y_2$ where $Y_1 \sim \chi^2_{k-1}$ and $Y_2 \sim \chi^2_{2(k-1)}$, but Y_1 and Y_2 are not necessarily independent. Hence, if we reject H_{07} when $-2\log\lambda > K_1 + K_2$ where $K_1 = \chi^2_{k-1;\alpha/2}$ and $K_2 = \chi^2_{2(k-1);\alpha/2}$ where $\chi^2_{n;\alpha}$ denotes the upper $100\alpha\%$ of χ^2_n, then it follows that <u>asymptotically</u> the proposed test

procedure has size less than or equal to α. As mentioned in the introduction, the asymptotic null distribution of $-2\log\lambda$ is difficult to obtain in this case.

CHAPTER FIVE

MAXIMUM LIKELIHOOD ESTIMATION FOR THE WITHOUT REPLACEMENT CASE

5.1 Introduction

In this chapter we consider maximum likelihood estimation of location parameters and failure rates of two parameter exponential under Type I censoring when sampling is done without replacement.

The layout of this chapter is as follows. Section 5.2 considers the one sample problem. In this case, MLE's of the location parameter and the failure rate are given in Bain (1978). For the location parameter, we have proposed a modified MLE which has asymptotically smaller mean squared error (MSE) than the MLE. Indeed, it is shown that the modified MLE achieves asymptotically 50% risk reduction than the MLE. Asymptotic distributions of the MLE's of the location parameter and the failure rate, as well as asymptotic distribution of the modified MLE and the scale parameter are also obtained in this section.

The two sample problem is considered in Section 5.3. Several cases are considered including those where the location and/or the scale parameters of the two populations are equal. As in the one sample case, modified MLE's achieve asymptotically 50% risk reduction for estimating the location parameters than the corresponding MLE's.

5.2 Estimation In The One Sample Case

Suppose that n items are put to test, and the lifetimes of these items are i.i.d. with common pdf

$$f(x) = \zeta\exp[-\zeta(x-\eta)]I_{[x>\eta]}, \tag{5.2.1}$$

where $I_A = 1$ if A happens, and $I_A = 0$ otherwise. The duration of the experiment is fixed, and is denoted by t. It is assumed that $\eta < t$, since otherwise there are no failures. Also an item which fails before the termination time is not replaced. Then, as explained in Section 1.1, the joint pdf of the ordered failure times and R is given by (1.1.7), namely

$$f(x_{(1)},\ldots x_{(r)},r) = \frac{n!}{(n-r)!}\,\zeta^r\exp[-\zeta\{\sum_{i=1}^{r}(x_{(i)}-\eta)+(n-r)(t-\eta)\}]$$

$$\times\, I_{[\eta<x_{(1)}\ldots<x_{(r)}<t]}$$

for $r = 1,2,\ldots n$ and

$$P(R = 0) = \exp[-n\zeta(t-\eta)]$$

It is clear from (1.1.7) that the MLE's of η and ζ are given respectively by

$$\hat{\eta} = X_{(1)}I_{[\eta<X_{(1)}<t]} + tI_{[X_{(1)}> t]},$$

$$\hat{\zeta} = [R/\{\Sigma_{i=1}^{R}(X_{(i)}-\hat{\eta}) + (n-R)(t-\hat{\eta})\}]I_{[R>1]}. \tag{5.2.2}$$

Note that using (1.1.6), the conditional pdf of $\hat{\eta}$ given $R = r$ where $r > 0$ is given by

$$f(z|r) = \frac{r\zeta\{\exp[-\zeta(z-\eta)] - \exp[-\zeta(t-\eta)]\}^{r-1}\exp[-\zeta(z-\eta)]}{(1 - \exp[-\zeta(t-\eta)])^{r}},$$

$$\text{for } \eta < z < t \qquad (5.2.3)$$

Also, for r=0

$$P(Z=t|r=0) = 1. \qquad (5.2.4)$$

Then, since we know from Section 1.1 that marginally $R \sim \text{Bin}(n, 1-\exp[-\zeta(t-\eta)])$, it follows using (5.2.3) and (5.2.4) that $\hat{\eta}$ has marginal pdf

$$f(z) = n\zeta\exp[-n\zeta(z-\eta)] \text{ if } \eta < z < t; \qquad (5.2.5)$$

$$P(Z=t) = \exp[-n\zeta(t-\eta)] \qquad (5.2.5a)$$

Hence $\hat{\eta}$ has the same pdf whether we are sampling with or without replacement. So that as proven in Section 3.2

$$n(\hat{\eta}-\eta) \xrightarrow{L} U \text{ as } n \to \infty, \qquad (5.2.6)$$

where U has an exponential distribution with failure rate ζ and

$$E[n^2(\hat{\eta}-\eta)^2] \to 2\zeta^{-2} \text{ as } n \to \infty. \qquad (5.2.7)$$

Also $\hat{\eta} \xrightarrow[a.s.]{P} \eta$ as $n \to \infty$.

Next we motivate the modified MLE. When ζ is known, $X_{(1)}$ is complete sufficient for η, and the UMVUE of η is given by $\hat{\eta} - (n\zeta)^{-1}I_{[\eta<X_{(1)}<t]}$. Substituting the estimator $\hat{\zeta}$ for ζ and noting that $I_{[\eta<X_{(1)}<t]} = I_{[R>1]}$, we propose the modified MLE of η as

$$\hat{\hat{\eta}} = \hat{\eta} - (n\hat{\zeta})^{-1}I_{[R>1]}$$

$$= \hat{\eta} - \{\Sigma_{i=1}^{R}(X_{(i)}-\hat{\eta}) + (n-R)(t-\hat{\eta})\}(nR)^{-1}I_{[R>1]} \qquad (5.2.8)$$

where 0/0 is interpreted as zero. Next writing $X_1,\ldots,X_n$ for the uncensored lifetimes of the n components, one gets $R = \Sigma_{i=1}^{n} I_{[X_i<t]}$. Also,

$$\Sigma_{i=1}^{R}(X_{(i)}-\hat{\eta}) + (n-R)(t-\hat{\eta})$$

$$=\Sigma_{i=1}^{n} X_i I_{[X_i<t]} + t\,\Sigma_{i=1}^{n} I_{[X_i>t]} - n\hat{\eta}$$

$$= \Sigma_{i=1}^{n} Y_i I_{[Y_i<t-\eta]} + (t-\eta)\Sigma_{i=1}^{n} I_{[Y_i>t-\eta]} - n(\hat{\eta}-\eta) \qquad (5.2.9)$$

where $Y_i = (X_i - \eta)$'s are iid exponential with failure rate ζ. Recall that $p = 1-\exp(-\zeta(t-\eta))$. Now, using the strong law of large numbers, as $n \to \infty$, one gets

$$n^{-1}\Sigma_{i=1}^{n} Y_i I_{[Y_i<t-\eta]} \xrightarrow{a.s.} E[Y_1 I_{[Y_1<t-\eta]}] = -(t-\eta)(1-p) + p\zeta^{-1}; \qquad (5.2.10)$$

$$n^{-1}\Sigma_{i=1}^{n} I_{[Y_i>t-\eta]} \xrightarrow{a.s.} P(Y_1>t-\eta) = 1-p; \qquad (5.2.11)$$

$$R/n \xrightarrow{P} p. \qquad (5.2.12)$$

Since $\hat{\eta} \xrightarrow{a.s.} \eta$, and $I_{[R>1]} \xrightarrow{a.s.} 1$ as $n \to \infty$ it follows from (5.2.8) - (5.2.12) that as $n \to \infty$,

$$\hat{\zeta} \xrightarrow{P} \zeta \qquad (5.2.13)$$

Now from (5.2.6), (5.2.8) and (5.2.13), one gets

$$n(\hat{\hat{\eta}}-\eta) = n(\hat{\eta}-\eta) - \hat{\zeta}^{-1} I_{[R>1]} \xrightarrow{L} U-\zeta^{-1}. \qquad (5.2.14)$$

Next we show that

$$E[n^2(\hat{\hat{\eta}} - \eta)^2] \longrightarrow E(U-\zeta^{-1})^2 = \zeta^{-2}. \qquad (5.2.15)$$

In view of (5.2.14) it suffices to show that $n^2(\hat{\hat{\eta}} - \eta)^2$ is uniformly integrable (u.i.) in $n > 1$. It was proven in Section 3.2

that $n^2(\hat{\eta} - \eta)^2$ is u.i. in $n \geq 1$. Hence, to prove (5.2.15), it suffices to show that $\hat{\zeta}^{-2} I_{[R \geq 1]}$ is u.i in $n \geq 1$. However, from (5.2.2),

$$\begin{aligned}
\hat{\zeta}^{-2} I_{[R\geq 1]} &= \{\Sigma_{i=1}^{R}(X_{(i)}-\hat{\eta}) + (n-R)(t-\hat{\eta})\}^2 R^{-2} I_{[R\geq 1]} \\
&\leq \{\Sigma_{i=1}^{R}(X_{(i)}-\eta) + (n-R)(t-\eta)\}^2 R^{-2} I_{[R\geq 1]} \\
&= \{\Sigma_{i=1}^{n}(X_i-\eta) I_{[X_i-\eta\leq t-\eta]} + (t-\eta)\Sigma_{i=1}^{n} I_{[X_i-\eta>t-\eta]}\}^2 \\
&\qquad \times R^{-2} I_{[R\geq 1]} \qquad (5.2.16)
\end{aligned}$$

Hence, for $0 < \zeta < 1$,

$$\begin{aligned}
&E[\hat{\zeta}^{-(2+\delta)} I_{[R\geq 1]}] \\
&= E[\{\Sigma_{i=1}^{n} Y_i I_{[Y_i\leq t-\eta]} + (t-\eta)\Sigma_{i=1}^{n} I_{[Y_i>t-\eta]}\}^{2+\delta} \\
&\quad \times R^{-(2+\delta)} I_{[R\geq 1]}] \\
&\leq E^{1/2}\{n^{-1}\Sigma_{i=1}^{n}(Y_i I_{[Y_i\leq t-\eta]} + (t-\eta) I_{[Y_i>t-\eta]})\}^{4+2\delta} \\
&\quad \times E^{1/2}[(n/R)^{4+2\delta} I_{[R\geq 1]}] \qquad (5.2.17) \\
&\leq E^{1/2}\{n^{-1}\Sigma_{i=1}^{n}((t-\eta) I_{[Y_i\leq t-\eta]} + (t-\eta) I_{[Y_i>t-\eta]})\}^{4+2\delta} \\
&\quad \times E^{1/2}[n/R]^{4+2\delta} I_{[R\geq 1]} \\
&= E^{1/2}\{n^{-1}(n(t-\eta))\}^{4+2\delta} \cdot E^{1/2}(n/R)^{4+2\delta} I_{[R\geq 1]} \qquad (5.2.17a)
\end{aligned}$$

Note that for every ε in (0,1) and $p = 1-\exp[-\zeta(t-\eta)]$

$$\begin{aligned}
E[n/R]^{4+2\delta} I_{[R\geq 1]} &\leq n^{4+2\delta} E(R^{-(4+2\delta)}(I_{[1\leq R\leq n\varepsilon p]} + I_{[R>n\varepsilon p]})) \\
&\leq n^{4+2\delta} P(R\leq n\varepsilon p) + (\varepsilon p)^{-(4+2\delta)} . \qquad (5.2.18)
\end{aligned}$$

Next observe that for every positive integer m,

$P(R<n\varepsilon p)<P(|R-np|>np(1-\varepsilon))<E|R-np|^{2m}/(np(1-\varepsilon))^{2m}=O(n^{-m})$. (5.2.19)

Choose $m > 6$ so that from (5.2.18) and (5.2.19) one gets

$$E[(n/R)^{4+2\delta}I_{[R>1]}] = O(1). \tag{5.2.20}$$

Now from (5.2.17), (5.2.17a) and (5.2.20),

$$\sup_{\eta>1} E[\hat{\zeta}^{-(2+\delta)}I_{[R>1]}] < \infty, \tag{5.2.21}$$

which proves the u.i. of $E[\hat{\zeta}^{-2}I_{[R>1]}]$. The proof of (5.2.15) is now complete. Note that similar calculations give $E[n(\hat{\hat{\eta}}-\eta)] \to 0$ as $n \to \infty$.

We find next the asymptotic distribution of $\hat{\zeta}^{-1}$. First, using (5.2.2) and (5.2.8), one gets $\sqrt{n}\ (\hat{\zeta}^{-1} - \zeta^{-1})$

$$= \sqrt{n}\ [\Sigma_{i=1}^{R}(X_{(i)}-\hat{\eta}) + (n-R)(t-\hat{\eta})-R\zeta^{-1}]R^{-1}I_{[R>1]} + \sqrt{n}\ \zeta^{-1}(I_{[R>1]}-1)$$

$$= n^{-1/2}\Sigma_{i=1}^{n}\{Y_iI_{[Y_i<t-\eta]} + (t-\eta)I_{[Y_i>t-\eta]}-\zeta^{-1}I_{[Y_i<t-\eta]}\}(n/R)I_{[R>1]}$$

$$- \sqrt{n}\{n(\hat{\eta}-\eta)\}R^{-1}I_{[R>1]} + \sqrt{n}\zeta^{-1}(I_{[R>1]}-1) \tag{5.2.22}$$

Since, $n(X_{(1)}-\eta) \xrightarrow{L} U$, $R/n \xrightarrow{a.s.} p$, and $I_{[R>1]} \xrightarrow{a.s.} 1$, one gets

$$\sqrt{n}\ \{n(X_{(1)}-\eta)\}R^{-1}I_{[R>1]} \xrightarrow{P} 0 \text{ as } n \to \infty. \tag{5.2.23}$$

and $\sqrt{n}\ \zeta^{-1}(I_{[R>1]}-1) \xrightarrow{P} 0$ as $n \to \infty$.

Write $W_i = Y_i\ I_{[Y_i<t-\eta]} + (t-\eta)I_{[Y_i>t-\eta]}-\zeta^{-1}I_{[Y_i<t-\eta]}$. Since the W_i's are i.i.d. with $E(W_1) = 0$ and $V(W_i) = p/\zeta^2$, using the central limit theorem and Slutsky's, it follows from (5.2.22),

(5.2.23) that

$$\sqrt{n}(\hat{\zeta}^{-1}-\zeta^{-1}) = \sqrt{n}^{-1/2}(\Sigma_{i=1}^{n}W_i)\,\frac{n}{R}\,I_{[R>1]} - \sqrt{n}(n(\hat{\eta}-\eta))R^{-1}I_{[R>1]}$$
$$+ \sqrt{n}\,\zeta^{-1}(I_{[R>1]}-1)$$

$$= \sqrt{n}\,(\bar{W}_n)\,\frac{n}{R}\,I_{[R>1]} - \sqrt{n}(n(\hat{\eta}-\eta))R^{-1}I_{[R>1]}$$
$$+ \sqrt{n}\zeta^{-1}(I_{[R>1]}-1)$$
$$\xrightarrow{L} N(0,1/p\zeta^2) \text{ as } n \to \infty. \tag{5.2.24}$$

where we have written $\bar{W}_n = \Sigma_{i=1}^{n}W_i/n$

Using Lemma 3.2.2, one now concludes that

$$\sqrt{n}\,(\hat{\zeta}-\zeta) \xrightarrow{L} N(0,p^{-1}\zeta^2) \text{ as } n \to \infty. \tag{5.2.25}$$

Next we show that

$$E[n(\hat{\zeta}^{-1} - \zeta^{-1})^2] \to p^{-1}\zeta^{-2} \text{ as } n \to \infty. \tag{5.2.26}$$

To prove (5.2.26), recall the definition of W_i after (5.2.23). Then, from (5.2.22) and using the C_δ-inequality for $\delta>0$, one gets

$$E(\sqrt{n}\,(\hat{\zeta}^{-1} - \zeta^{-1}))^{2+\delta} \le 3^{1+\delta}[E(\frac{n^{2+\delta}}{R^{2+\delta}}\,I_{[R>1]}\{(n^{-1/2}(\Sigma_{i=1}^{n}W_i))^{2+\delta}$$

$$+ (\sqrt{n}\,(\hat{\eta}-\eta))^{2+\delta}\}) + E(\sqrt{n}\,\zeta^{-1}(I_{[R>1]}-1))^{2+\delta}]$$

$$\le 3^{1+\delta}[E^{1/2}[(\frac{n}{R})^{4+2\delta}I_{[R>1]}]E^{1/2}[n^{-1/2}(\Sigma_{i=1}^{n}W_i)]^{4+2\delta}$$

$$+ E^{1/2}[(\frac{n}{R})^{4+2\delta}I_{[R>1]}]E^{1/2}[n^{1/2}(\hat{\eta}-\eta)]^{4+2\delta}]$$

$$+ E(\sqrt{n}\,\zeta^{-1}(I_{[R>1]}-1))^{2+2\delta}]\,. \tag{5.2.27}$$

From (5.2.20) we know that $E(n/R)^{4+2\delta}I_{[R>1]} = O(1)$. Also,

since

$E|W_1|^s < \infty$ for $s > 0$, using Lemma 3.2.1 one gets

$$n^{-(2+\delta)}E(\Sigma_{i=1}^{n}W_i)^{4+2\delta} \leq n^{-(2+\delta)}K\ n^{2+\delta} = O(1) \qquad (5.2.28)$$

Furthermore, as $n \to \infty$

$$n^{-(2+\delta)}E\big(n(\hat{\eta}-\eta)\big)^{4+2\delta} \leq n^{-(2+\delta)}\int_0^\infty z^{4+2\delta}e^{-z}dz = O\big(n^{-(2+\delta)}\big) \qquad (5.2.29)$$

$$\text{Also } E\big(\sqrt{n}\zeta(I_{[R>1]}-1)\big)^{2+2\delta} = n^{1+\delta}\zeta^{2+2\delta}P(R=0)$$

$$= n^{1+\delta}\zeta^{2+2\delta}\exp[-n\zeta(t-\eta)] \to 0 \qquad (5.2.30)$$

It follows that the right hand side of (5.2.27) is O(1). This proves the uniform integrability of $n(\hat{\zeta}^{-1}-\zeta^{-1})^2$ and together with (5.2.24) proves (5.2.26).

Remark 1. If ζ is known then the modified MLE is the UMVUE estimator. If η is known then we are again in the regular exponential family of densites and we can use Fisher's theorem directly to obtain the asymptotic behavior of the M.L.E. Also u.i. results are still true with minor modifications to the proofs given here.

Remark 2. The search for uniformly minimum variance unbiased estimators of parameters of interest becomes quite formidable when sampling is done without replacement, due to the complexity of the distribution of the sufficient statistic $(X_{(1)}, R, \Sigma_{i=1}^{R}, X_{(i)})$. Also the family of distributions induced by these statistics is possibly not complete due to the fact that the minimal sufficient statistics is of dimension 3, while the parameter of interest has dimension 2. However, by using the joint density of the ordered

failure times and R and arguments similar to those used in Theorem 2.2.1, one can conclude that a function $h(\eta,\zeta)$ is estimable only if it is of the form $\Sigma_{j=0}^{\infty} u_j(\eta)\zeta^j$ where $u_0(\eta)$ does not depend on η. Hence when both η and ζ are unknown neither η nor ζ^{-1} admit an unbiased estimator based on any function of the ordered failure times and R. Therefore, we cannot have an unbiased estimator based on $(X_{(1)}, R, \Sigma_{i=1}^{R} X_{(i)})$ either. Also, if ζ is known only $X_{(1)}$ is complete sufficient for η and using the Rao-Blackwell-Lehmann-Scheffe Theorem, $h(X_{(1)}) = X_{(1)} - \zeta^{-1}n^{-1}[1-I_{[X_{(1)}> t]}]$ is the UMVUE of η

If η is known, then $(R, \Sigma_{i=1}^{R} X_{(i)})$ is sufficient for ζ. Because we are now in the regular exponential family, we know that the family of densities induced by $(R, \Sigma_{i=1}^{R} X_{(i)})$ is not complete. An argument similar to the one used when both parameters are unknown, shows that ζ^{-1} is not estimable in this case either.

5.3 Estimation in the Two Sample Case

Suppose now that two independent sets of items are put to test, where the first set contains n_1 elements, and the second set contains n_2 elements. As before, denote by $X_{i1},\ldots,X_{in_i}$ the lifetimes of the n_i items for the i^{th} set $(i=1,2)$. The X_{ij}'s are all assumed to be independent and $X_{i1},\ldots,X_{in_i}$ are assumed to be i.i.d. with common pdf

$$f(x) = \zeta_i \exp[-\zeta_i(x-\eta_i)]\, I_{[x>\eta_i]} \quad (i=1,2); \tag{5.3.1}$$

Again, the duration of the experiment is fixed, and the censoring

times for the two sets are denoted by t_1 and t_2. It is assumed that $\eta_i < t_i$ (i=1,2). Also, for definiteness let $t_1 < t_2$. In this case, since censoring is done without replacement, any item failing before the censoring time is neither repaired nor replaced. Denote by R_i the number of failures for the i^{th} set before time t_i then $R_i \sim \text{Bin}(n_i, 1 - \exp[-\zeta_i(t_i-\eta_i)])$ ($i = 1,2$). For $R_i = r_i(>0)$ let $X_{(i1)} < \ldots < X_{(ir_i)}$ denote the ordered failure times for the i^{th} set. Generalizing (1.1.7), the joint pdf of $X_{(i1)},\ldots,X_{(iR_i)}, R_i$ (i=1,2) is given by

$$f(x_{(11)},\ldots,x_{(1r_1)},r_1,x_{(21)},\ldots,x_{(2r_2)},r_2)$$

$$= \Pi_{i=1}^2[\{n_i!/(n_i-r_i)!\}\zeta_i^{r_i}]\exp[-\Sigma_{i=1}^2\zeta_i\{\Sigma_{j=1}^{r_i}(x_{(ij)}-\eta_i)$$

$$+ (n_i-r_i)(t_i-\eta_i)\}]\times\prod_{i=1}^{2} I_{[\eta_i<x_{(i1)}<\ldots<x_{(ir_i)}<t_i]}, \tag{5.3.2}$$

when $r_1 > 1$, and $r_2 > 1$.

$$f(x_{(11)},\ldots,x_{(1r_1)},r_1,0)$$

$$= \{n_1!\zeta_1^{r_1}/(n_1-r_1)!\}\exp[-\zeta_1\{\Sigma_{j=1}^{r_1}(x_{(ij)}-\eta_1)+(n_1-r_1)(t_1-\eta_1)\}$$

$$- n_2\zeta_2(t_2-\eta_2)]I_{[\eta_1<x_{(11)}<\ldots<x_{(1r_1)}<t_1]} \tag{5.3.3}$$

when $r_1 > 1$;

$$f(0,x_{(21)},\ldots,x_{(2r_2},r_2)$$

$$= \{n_2!\zeta_2^{r_2}/(n_2-r_2)!\}\exp[-n_1\zeta_1(t_1-\eta_1)-\zeta_2\{\Sigma_{j=1}^{r_2}(x_{(2j)}-\eta_2)$$

when $r_2 \geq 1$;

$$f(0,0) = \exp\,[-\Sigma_{i=1}^{2} n_i \zeta_i (t_i - \eta_i)]. \tag{5.3.5}$$

First, consider the case when η_1, η_2 ζ_1 and ζ_2 are all distinct. Then, direct generalization of (5.2.2) gives the MLEs of η_i's and ζ_i's as $\hat{\eta}_i = X_{(i1)}(1-I_{[X_{(i1)}>t]}) + t_i I_{[X_{(i1)}>t_i]}$

$$\hat{\zeta}_i = [R_i/(\Sigma_{j=1}^{R_i}\{X_{(ij)}-X_{(i1)}\}+(n_i-R_i)(t_i-\eta_i))]I_{[R_i>1]} \tag{5.3.6}$$

for $i = 1,2$.

The modified MLEs of η_i's are given by

$$\hat{\hat{\eta}}_i = \hat{\eta}_i - (n_i\hat{\zeta}_i)^{-1}(1-I_{[X_{(i1)}>t_i]}) \ (i=1,2) \tag{5.3.7}$$

The properties of $\hat{\eta}$, $\hat{\hat{\eta}}$, $\hat{\zeta}^{-1}$ and $\hat{\zeta}$ derived in the one sample case extend immediately to their two sample analogues.

Next we consider the case when $\eta_1 = \eta_2 = \eta$, but ζ_1 and ζ_2 need not be the same. In this set up estimators of η are given in Ghosh and Razampour (1984) in the uncensored case and by Chiou and Cohen (1984) in the Type II censored case. In this case writing $Z = \min(X_{(11)}, X_{(21)})$, the MLEs of η, ζ_1 and ζ_2 are given respectively by

$$\hat{\eta} = Z[1-I_{[Z>t_1]}] + t_1 I_{[Z>t_1]}; \tag{5.3.8}$$

$$\hat{\zeta}_i = [R_i/\{\Sigma_{j=1}^{R_i}(X_{(ij)}-\hat{\eta}) + (n_i-R_i)(t_i-\hat{\eta})\}]I_{[R_i>1]} \tag{5.3.9}$$

for $i = 1,2$.

It is easy to verify using (5.2.5) and (5.2.5a) that $\hat{\eta}$ has pdf

$$f(u) = a\exp[-a(u-\eta)] \qquad \eta < u < t_1;$$

$$P(U=t_1) = \exp[-a(t_1-\eta)] \tag{5.3.10}$$

where $a = n_1\zeta_1 + n_2\zeta_2$. Also, from (5.3.10) it is easy to check, via the Borel-Cantelli lemma that $\hat{\eta} \xrightarrow{a.s.} \eta$ as $\min(n_1,n_2) \to \infty$. As in Chapter Three, to find the asymptotic distribution of $\hat{\eta}$, first let $n = n_1 + n_2$. Assume that

$$\lim_{n\to\infty} n_1/n = \lambda, \qquad 0 < \lambda < 1 \tag{5.3.11}$$

Also, Theorem 3.3.1 in Chapter Three asserts that if (5.3.11) holds,

$$n(\hat{\eta}-\eta) \xrightarrow{L} U, \tag{5.3.12}$$

where U is exponential with failure rate $g = \lambda\zeta_1 + (1-\lambda)\zeta_2$, and also direct calculations in (3.3.6) give

$$E\left(n^2(\hat{\eta}-\eta)^2\right) \to 2g^{-2} \quad \text{as } n \to \infty. \tag{5.3.13}$$

Actually in (3.3.10) it is shown that

$$n^2(\hat{\eta}-\eta)^2 \text{ is u.i. in n.} \tag{5.3.14}$$

To motivate the modified MLE, note that if ζ_1 and ζ_2 are known, then Z is complete sufficient for η and the UMVUE of η is given by

$$\hat{\eta} - a^{-1}[1-I_{[Z>t_1]}].$$

Thus, when ζ_1 and ζ_2 are unknown, we propose the modified ML estimator of η as

$$\tilde{\tilde{\eta}} = \hat{\eta} - \hat{a}^{-1}[1-I_{[Z>t_1]}] \tag{5.3.15}$$

where $\hat{a}$ is obtained by plugging the ML estimators $\hat{\zeta}_1$ and $\hat{\zeta}_2$ for ζ_1 and ζ_2 in a. Under the assumption (5.3.11) and writing

$$n\hat{a}^{-1} = n \left(\frac{n_1 R_1}{\sum_{j=1}^{R_1}(X_{(1j)}-\hat{\eta})+(n_1-R_1)(t_1-\hat{\eta})}\right. \quad I_{[R_1>1]}$$

$$+ \frac{n_2 R_2}{\Sigma_{j=1}^{R_2}(X_{(2j)}-\hat{\eta})+(n_2-R_2)(t_2-\hat{\eta})} I_{[R_2>1]})^{-1}$$

where

$$\frac{\Sigma_{j=1}^{R_i}(X_{(ij)}-\hat{\eta}) + (n_i-R_i)(t_i-\hat{\eta})}{n_i}$$

$$= n_i^{-1} \sum_{i=1}^{n} [(X_{ij}-\eta) I_{[X_{ij}-\eta<t_i-\eta]}$$

$$+ (t_i-\eta) I_{[X_{ij}-\eta>t_i-\eta]}] + (\hat{\eta}-\eta) \xrightarrow{a.s.} p_i \zeta_i^{-1}$$

and $R_i/n_i \xrightarrow{a.s.} p_i$ by the SLLN ($i=1,2$). Hence it is easy to see that $n\hat{a}^{-1} \xrightarrow{a.s.} (\lambda\zeta_1+(\eta-\lambda)\zeta_2)^{-1} = g^{-1}$. Also $(1 - I_{[Z>t_1]}) = I_{[\eta<Z<t_1]} \xrightarrow{a.s.} 1$. Hence, from (5.3.12) and (5.3.15), one gets

$$n(\tilde{\eta}-\eta) \xrightarrow{L} U - g^{-1}. \tag{5.3.16}$$

In view of the fact that $(1-I_{[Z>t_1]}) = I_{[\eta<Z<t_1]}$, (5.3.14) and the inequality

$$n^2(\tilde{\eta} -\eta)^2 < 2[n^2(\hat{\eta}-\eta)^2 + n^2\hat{a}^{-2} I_{[\eta<Z<t_1]}], \tag{5.3.17}$$

for proving the uniform integrability of $n^2(\tilde{\eta}-\eta)^2$ in n, it suffices to show that $n^2\hat{a}^{-2} I_{[\eta<Z<t_1]}$ is uniformly integrable in n. (5.3.18)

This will then imply that

$$E[n^2(\hat{\eta}-\eta)]^2 \to g^{-2} \text{ as } n \to \infty. \tag{5.3.19}$$

so that comparing (5.3.13) and (5.3.19) it follows that $\tilde{\eta}$ achieves asymptotically 50% MSE reduction than $\hat{\eta}$.

To prove (5.3.18), first notice that

$$E(n^2\hat{a}^{-2})^{1+\frac{\delta}{2}} I_{[\eta<Z<t_1]} < \Sigma_{i=1}^2 (n/n_i)^{2+\delta} E(\hat{\zeta}_i^{-(2+\delta)}) I_{[R_i>1]}. \tag{5.3.20}$$

Since (5.3.11) holds, from (5.3.20) it follows that it suffices to show

$$\sup_{n>1} E(\hat{\zeta}_i^{-(2+\delta)} I_{[R_i>1]}) < \infty \text{ for } i=1,2 \tag{5.3.21}$$

to prove (5.3.18). But the proof of (5.3.21) is accomplished much the same as (5.2.17) - (5.2.20).

To find the asymptotic distribution of $\hat{\zeta}_1^{-1}$ and $\hat{\zeta}_2^{-1}$, using (5.3.9) first write

$$\begin{aligned}
&\sqrt{n_i}(\hat{\zeta}_i^{-1} - \zeta_i^{-1}) \\
&\quad = (\sqrt{n_i}/R_i)I_{[R_i>1]}[\Sigma_{j=1}^{n_i}(X_{ij}-\zeta_i^{-1})I_{[X_{ij}<t_1]} + \Sigma_{j=1}^{n_i} t_i I_{[X_{ij}>t_i]} \\
&\qquad\qquad - n_i\hat{\eta}] + \sqrt{n}_i\zeta_i^{-1}(I_{[R_i>1]} - 1) \\
&\quad = (n_i/R_i)I_{[R_i>1]}[n_i^{-1/2}\Sigma_{j=1}^{n_i}\{(X_{ij}-\eta-\zeta_i^{-1})I_{[X_{ij}<t_i]} \\
&\qquad\qquad + (t_i-\eta)I_{[X_{ij}>t_i]}\}] \\
&\qquad + \sqrt{n}_i\zeta_i^{-1}(I_{[R_i>1]} - 1) \\
&\qquad - (n_i^{1/2}/R_i)I_{[R_i>1]}\, n_i(\hat{\eta}-\eta), \quad i = 1,2.
\end{aligned} \tag{5.3.22}$$

In view of (5.3.11), (5.3.12) and the facts that $R_i/n_i \xrightarrow{a.s.} p_i$ as $n \to \infty$ and $I_{[R_i>1]} \xrightarrow{a.s.} 1$ as $\min(n_1,n_2) \to \infty$, one gets

$$(n_i^{1/2}/R_i)I_{[R_i>1]}n_i(\hat{\eta}-\eta) \xrightarrow{P} 0 \text{ as } \min(n_1,n_2) \to \infty, \tag{5.3.23}$$

and $\sqrt{n_i}\ \zeta_i^{-1}(I_{[R_i>1]}-1) \xrightarrow{P} 0$ as $n \to \infty$ for $i = 1,2$.

Also, let

$$W_{ij} = n_i^{-1/2}\Sigma_{j=1}^{n_i}\{(X_{ij}-\eta-\zeta_i^{-1})I_{[X_{ij}<t_i]} + (t_i-\eta_i)I_{[X_{ij}>t_i]}\}$$

for $i = 1,2$. $j = 1\ldots n_i$, so that, $W_{i1},\ldots W_{in_i}$ are i.i.d. with $EW_{i1} = 0$ and $V(W_{i1}) = p_i\zeta_i^{-2}$ for $i = 1,2$.

It follows using (5.3.22), the CLT and calculations similar to (5.2.24) that as $\min(n_1,n_2) \to \infty$

$$\sqrt{n_i}\ (\hat{\zeta}_i^{-1} - \zeta_i^{-1}) \xrightarrow{L} N(0,(p_i\zeta_i^2)^{-1}) \text{ for } i = 1,2 \tag{5.3.24}$$

and via Lemma 3.2.2, one now concludes that

$$\sqrt{n_i}\ (\hat{\zeta}_i - \zeta_i) \xrightarrow{L} N(0,p_i^{-1}\zeta_i^2).$$

Also, using inequalities similar to (5.2.27) and (5.2.28), one can prove the uniform integrability of $n_i(\hat{\zeta}_i^{-1} - \zeta_i^{-1})^2$ as $\min(n_1,n_2) \to \infty$, for $i = 1,2$, and conclude that

$$E[n_i(\hat{\zeta}_i^{-1} - \zeta_i^{-1})^2] \to (p_i\zeta^2)^{-1} \text{ as } \min(n_1,n_2) \text{ as } n \to \infty. \tag{5.3.25}$$

Remark 1 If at least one failure rate is known all convergence and u.i. results still hold true with minor modifications to the proofs given here. The modified MLE in this situation would be obtained by substituting the known failure rate in the expression for $\hat{\eta}$, instead of its MLE.

Remark 2 If η is known, we can invoke Fisher's theorem to conclude asymptotic normality of $\sqrt{n_i}(\hat{\zeta}_i^{-1} - \zeta_i^{-1})$. The u.i. property of $(\sqrt{n_i}(\hat{\zeta}^{-1} - \zeta^{-1}))^2$ when η is known is obvious from the previous argument. Also it is then straightforward to see that

$E(n(\hat{\eta}-\eta)) \to 1$ while $E(n(\tilde{\eta}-\eta)) \to 0$. Hence the modified MLE attains 100% bias reduction over the usual MLE.

Remark 3. In this situation it can be easily seen using (5.3.2) - (5.3.5) that $\underset{\sim}{T}_1 = (Z, R_1, S_1, R_2, S_2)$($Z = \min(X_{(11)}, X_{(21)})$ and $S_i = \Sigma_{j=1}^{R_i} X_{(ij)}$ for $i = 1,2$) is sufficient for (η, ζ_1, ζ_2).

Because the density of $\underset{\sim}{T}_1$ is very difficult to obtain and may not be complete, uniformly minimum variance unbiased estimators for parameters of interest cannot be derived. However, by working with the joint density of ordered failure times for both groups, R_1 and R_2 and using arguments similar to those used in Theorem 2.3.5 one can show that a function $h(\eta, \zeta_1, \zeta_2)$ is estimable only if it has the form $\sum_{r_1=0}^{\infty} \sum_{r_2=0}^{\infty} u_{r_1,r_2}(\eta)\zeta_1^{r_1}\zeta_2^{r_2}$ where $u_{0,0}(\eta)$ does not depend on η.
Hence, neither η, ζ_1^{-1}, nor ζ_2^{-1} admits an unbiased estimator based on any function of ordered failure times for both groups, R_1 and R_2. Therefore there cannot exist an unbiased estimator of η, ζ_1^{-1} and ζ_2^{-1} based on $\underset{\sim}{T}_1$.

If one failure rate is known, say ζ_1, again using this argument, it can be shown that there is no unbiased estimator for ζ_2^{-1}. No conclusion can be made about estimating η.

If both failure rates are known, then η is estimable and has UMVUE given by $h(Z) = Z - a^{-1}[1 - I_{[Z>t_1]}]$ where $a = n_1\zeta_1 + n_2\zeta_2$.

If η is known, one can also show that neither ζ_1^{-1} or ζ_2^{-1} is estimable.

Next we consider the case when $\zeta_1 = \zeta_2 = \zeta$, but η_1 and η_2 are not necessarily equal. In this case, an examination of (5.3.2) - (5.3.5) reveals that $X_{(11)}$, $X_{(21)}$ and $R = R_1 + R_2$ is minimal sufficient for η_1, η_2 and ζ. Maximum likelihood estimators are given respectively by

$$\hat{\eta}_i = X_{(i1)}[1 - I_{[X_{(i1)}>t_i]}] + t_i I_{[X_{(i1)}>t_i]} \text{ for } i=1,2. \quad (5.3.26)$$

$$\hat{\zeta} = [R/\{\Sigma_{i=1}^2 \Sigma_{j=1}^{R_i}(X_{(ij)}-\hat{\eta}_i) + \Sigma_{i=1}^2(n_i-R_i)(t_i-\hat{\eta}_i)\}]I_{[R_1>1,R_2>1]}$$

$$+ [R_1/\{\Sigma_{j=1}^{R_1}(X_{(1j)}-\hat{\eta}_1) + (n_1-R_1)(t_1-\hat{\eta}_1)\}]I_{[R_1>1,R_2=0]}$$

$$+ [R_2/\{\Sigma_{j=1}^{R_2}(X_{(2j)}-\hat{\eta}_2) + (n_2-R_2)(t_2-\hat{\eta}_2)\}]I_{[R_1=0,R_2>1]} \quad (5.3.27)$$

In this case, the modified MLE's of η_i's are given by

$$\hat{\hat{\eta}}_i = X_{(i1)} - (n_i\hat{\zeta})^{-1}[1 - I_{[X_{(i1)}>t_i]}] \quad (i=1,2)$$

Then, the following results are true under (5.3.11) as $n \to \infty$.

(i) $n(\hat{\eta}_1-\eta_1) \xrightarrow{L} \lambda^{-1}U$ where U is exponential with failure rate ζ

<u>Proof</u>. For $0 < z < n_1(t_1-\eta_1)$

$$P(n(\hat{\eta}_1-\eta_1) > z) = P[X_{(11)} > z/n + \eta_1]$$

$$= \int_{z/n+\eta_1}^{t_1} n_1\zeta \exp[-n_1\zeta(x-\eta)]dx + \exp[-n_1\zeta(t_1-\eta_1)]$$

$$= \exp[-\frac{n_1}{n}\zeta z]$$

$$P(n(\hat{\eta}_1-\eta_1)) = n(t_1-\eta_1) = \exp[-n\zeta(t_1-\eta_1)]$$

Taking limits as $n \to \infty$, the result follows. $\square$

(ii) $E[n_1(\hat{\eta}_1-\eta_1)]^2 \to 2\zeta^{-2}$,

$E[n_1(\hat{\eta}_1-\eta_1)] \to \zeta^{-1}$ as $n_1 \to \infty$

Proof Straighforward calculations.

(iii) $n_2(\hat{\eta}_2-\eta_2) \to U$ as $n_2 \to \infty$.

Proof Along the same lines as (i).

(iv) $E(n_2(\hat{\eta}_2-\eta_2))^2 \to 2\zeta^{-2}$

$E(n_2(\hat{\eta}_2-\eta_2)) \to \zeta^{-1}$ as $n_2 \to \infty$.

Proof Straightforward calculations.

(v) $n_1(\hat{\hat{\eta}}_1-\eta_1) \xrightarrow{L} (U-\zeta^{-1})$ as $n_1 \to \infty$.

Proof Using the fact that $1-I_{[X_{(11)} > t_1]} = I_{[\eta<X_{(11)}<t_1]}$, one gets

$$n_1(\hat{\hat{\eta}}_1-\eta_1) = n_1(X_{(11)}-\eta_1) - (n_1\ \hat{\zeta})^{-1} I_{[\eta_1<X_{(11)}<t_1]}. \qquad (5.3.28)$$

In view of (i) and (5.3.28), it suffices to show that $\hat{\zeta}^{-1} \xrightarrow{P} \zeta^{-1}$.

It is easy to check that $\hat{\eta}_i \xrightarrow{a.s.} \eta_i \quad i = 1,2$.

Also, $T_i = \dfrac{\Sigma_{j=1}^{R_i}(X_{(ij)}-\hat{\eta}_i) + (n_i-R_i)(t_i-\hat{\eta}_i)}{n_i}$

$$= \frac{\Sigma_{j=1}^{n_i}[(X_{ij}-\eta_i)I_{[X_{ij}-\eta_i\le t_i-\eta_i]} + (t_i-\eta_i)I_{[X_{ij}-\eta_i>t_i-\eta_i]} - n_i(\hat{\eta}_i-\eta)}{n_i}$$

$\xrightarrow{a.s.} \zeta^{-1}p_i$ as $n \to \infty$ and $R_i/n_i \xrightarrow{P} p_i$ (for i=1,2) by the WLLN where $p_i = 1 - \exp[-\zeta(t_i-\eta_i)]$.

Note also, that $I_{[R_1 \geq 1,\ R_2 \geq 1]} \xrightarrow{a.s.} 1$

while $I_{[R_1=0, R_2 \geq 1]} \xrightarrow{a.s.} 0$ and $I_{[R_1 \geq 1,\ R_2=0]} \xrightarrow{a.s.} 0$.

Hence, using (5.3.27)

$$\hat{\zeta}^{-1} = [(\frac{n_1}{n}\frac{R_1}{n_1} + \frac{n_2}{n}\frac{R_2}{n_2} / \frac{n_1}{n}\frac{T_1}{n_1} + \frac{n_2}{n}\frac{T_2}{n})I_{[R_1 \geq 1,\ R_2 \geq 1]}$$

$$+ (\frac{n_1}{n}\frac{R_1}{n}/\frac{n_1}{n}\frac{T_1}{n_1})I_{[R_1 \geq 1,\ R_2=0]} + (\frac{n_2}{n}\frac{R_2}{n_2}/\frac{n_2}{n}\frac{T_2}{n_2})I_{[R_1=0,\ R_1>0]}]^{-1}$$

$$\xrightarrow{a.s.} [(\lambda p_1+(1-\lambda)p_2/\lambda\zeta^{-1}p_1+(1-\lambda)\zeta^{-1}p_2)]^{-1}$$

$$= [1/\zeta^{-1}]^{-1} = \zeta^{-1} \quad \text{as } n \to \infty. \qquad \square$$

(vi) $E[n(\hat{\hat{\eta}}_1-\eta_1)]^2 \to \lambda^{-2}\zeta^{-2}$ and $E[n(\hat{\hat{\eta}}_1-\eta_1)] \to 0$ as $n \to \infty$.

Proof: For $0 < \delta < 1$

$$[n(\hat{\hat{\eta}}_1-\eta_1)]^{2+\delta} \leq 2^{1+\delta}[n^{2+\delta}(X_{(11)}-\eta_1)^{2+\delta}$$

$$+ (\frac{n}{n_1})^{2+\delta}\ \hat{\zeta}^{-(2+\delta)}I_{[\eta_1 < X_{(11)} < t_1]}]$$

Note that

$$\sup_{n \geq 1} E(n^{2+\delta}(X_{(11)}-\eta_1)^{2+\delta}) \leq \sup_{n \geq 1}\int_{\eta_1}^{\infty} (n(u-\eta_1))^{2+\delta} n_1\zeta\exp[-n_1\zeta(u-\eta_1)]du$$

$$= \sup_{n \geq 1}(\frac{n}{n_1})^{2+\delta}\int_{\eta_1}^{\infty} z^{2+\delta}\zeta\exp[-\zeta z]dz = O(1)$$

while

$$E(\hat{\zeta}^{-(2+\delta)}I_{[\eta_1<X_{(11)}<t_1]} \leq E[\frac{T_1+T_2}{R}]^{2+\delta}I_{[R_1 \geq 1,\ R_2 \geq 1]}$$

$$+ E[\frac{T_1}{R_1} I_{[R_1>1]}]^{2+\delta} + E[\frac{T_2}{R_2} I_{[R_2>1]}]^{2+\delta}$$

$$< [\frac{n_1(t_1-\eta_1) + n_2(t_2-\eta_2)}{n_1}]^{2+\delta} \; E[\frac{n_1}{R_1}]^{2+\delta} I_{[R_1>1]} + 0(1)$$

$$= 0(1)$$

where we have used arguments similar to (5.2.16) - (5.2.20).

Hence $(n_1(\hat{\hat{\eta}}_1-\eta_1))^2$ is u.i and using (v) the result follows. □

(vii) $n_2(\hat{\hat{\eta}}_2-\eta_2) \xrightarrow{L} (U-\zeta^{-1})$.

Proof The proof follows along the lines given in (v).

(viii) $E(n_2(\hat{\hat{\eta}}_2-\eta_2))^2 \to \zeta^{-2}$ and $E(n_2(\hat{\hat{\eta}}_2-\eta_2)) \to 0$ as $n_2 \to \infty$.

Proof The proof follows along the lines given in (vi).

(ix) $\sqrt{n}\,(\hat{\zeta}^{-1}-\zeta^{-1}) \xrightarrow{L} N(0,\zeta^{-2}(\lambda p_1^{-1}+ (1-\lambda)p_2^{-1})(\lambda p_1 + (1-\lambda)p_2)^{-1})$

Proof

$\sqrt{n}\,(\hat{\zeta}^{-1} - \zeta^{-1}) =$

$$\sqrt{n}\,[\frac{T_1+T_2}{R_1+R_2} I_{[R_1>1,\ R_2>1]} - \zeta]$$

$$+ \sqrt{n}\,\frac{T_1}{R_1} I_{[R_1>1,\ R_2=0]} + \sqrt{n}\,\frac{T_2}{R_2} I_{[R_1=0,\ R_2>1]}$$

$$= n^{-1/2} \cdot \frac{n}{R} \Big[\sum_{i=1}^{2}\sum_{j=1}^{n_i}((X_{ij}-\eta_i)I_{[X_{ij}<t_j]}+(t_i-\eta_i)I_{[X_{ij}>t_i]}$$

$$- \zeta^{-1} I_{[X_{ij}-\eta_i<t_i-\eta_i]})\Big] I_{[R_1>1,\ R_2>1]}$$

$$- \frac{\sqrt{n}}{R}\,(n_1(\hat{\eta}_1-\eta_1)) I_{[R_1>1,\ R_2>1]} - \frac{\sqrt{n}}{R}\,(n_2(\hat{\eta}_2-\eta_2)) I_{[R_1>1,\ R_2>1]}$$

$$+ \sqrt{n}\,\frac{T_1}{R_1}\, I_{[R_1 \geq 1,\ R_2 = 0]} + \sqrt{n}\,\frac{T_2}{R_2}\, I_{[R_1 = 0,\ R_2 \geq 1]}$$

$$+ \sqrt{n}\,\zeta (I_{[R_1 \geq 1,\ R_2 \geq 1]} - 1) \qquad (5.3.29)$$

Since $n_1(\hat{\eta}_1 - \eta_1) \xrightarrow{L}$ exponential with failure rate ζ and $\frac{R}{n} \xrightarrow{a.s.} \lambda p_1 + (1-\lambda)p_2$ it follows that the second and third term in (5.3.29) tend to zero a.s.

Also $P[\sqrt{n}\,\frac{T_1}{R_1}\, I_{[R_1 \geq 1,\ R_2 = 0]} \neq 0] < P[R_2 = 0] = \exp[-n_2\zeta(t_2 - \eta_2)]$

Hence, using the Borel-Cantelli lemma $\sqrt{n}\,\frac{T_1}{R_1}\, I_{[R_1 \geq 1,\ R_2 = 0]} \xrightarrow{a.s.} 0$

Similarly, $\sqrt{n}\,\frac{T_2}{R_2}\, I_{[R_1 = 0,\ R_2 > 0]} \xrightarrow{a.s.} 0$

and $\sqrt{n}\,\zeta(I_{[R_1 \geq 1,\ R_2 \geq 1]} - 1) \xrightarrow{a.s.} 0$.

Then the limiting behavior for $\sqrt{n}\,(\hat{\zeta}^{-1} - \zeta^{-1})$ is the same as the limiting behavior of

$$n^{-1/2}\Big[\sum_{i=1}^{2} \sum_{j=1}^{n_i} [(X_{ij} - \eta_i) I_{[X_{ij} < t_j]} + (t_i - \eta_i) I_{[X_{ij} > t_i]}$$

$$- \zeta^{-1} I_{[X_{ij} - \eta_i < t_i - \eta_i]}]\Big] \frac{n}{R}\, I_{[R_1 \geq 1,\ R_2 \geq 1]}$$

$$= n^{-1/2}\Big[\sum_{j=1}^{n_1} Y_{1j} + \sum_{j=1}^{n_2} Y_{2j}\Big] \frac{n}{R}\, I_{[R_1 \geq 1,\ R_2 \geq 1]}$$

where

$$Y_{ij} = (X_{ij} - \eta_i) I_{[X_{ij} - \eta_i < t_i - \eta_i]} + (t_i - \eta_i) I_{[X_{ij} - \eta_i > t_i - \eta_i]}$$

$$- \zeta^{-1} I_{[X_{ij}-\eta_i < t_i-\eta_i]} \quad j=1,\ldots n_i$$

are iid with mean zero and variance $p_i\zeta^{-2}$ $(i=1,2)$.

Hence, using the central limit theorem as in Chapter Three, and the fact that $I_{[R_1\geq 1,\ R_2\geq 1]} \xrightarrow{a.s.} 1$ and $n/R \xrightarrow{P} (\lambda p_1+(1-\lambda)p_2)^{-1}$. (In fact it can be shown using the Borel-Cantelli lemma that $n/R \xrightarrow{a.s.} (\lambda p_1+(1-\lambda)p_2)^{-1}$).

$$\sqrt{n}\,(\hat{\zeta}^{-1}-\zeta^{-1}) \xrightarrow{L} N(0,\frac{\zeta^{-2}}{(\lambda p_1 + (1-\lambda)p_2)}).$$

Note that using Lemma 3.2.2 one gets that

$$\sqrt{n}\,(\hat{\zeta}-\zeta) \xrightarrow{L} N(0, \frac{\zeta^2}{\lambda p_1 + (1-\lambda)p_2}) \qquad \square$$

(x) $E(n(\hat{\zeta}^{-1}-\zeta^{-1})^2) \to \zeta^{-2}(\lambda p_1 + (1-\lambda)p_2)^{-1}$

Proof It suffices to show that

$$\sup_{n\geq 1} E[\sqrt{n}\,|\hat{\zeta}^{-1}-\zeta^{-1}|]^{2+2\delta} < \infty \text{ for some } 0 < \delta < 1 \tag{5.3.30}$$

To proof (5.3.30), note that

$$E(\sqrt{n}|\hat{\zeta}^{-1}-\zeta^{-1}|)^{2+2\delta} \leq 3^{1+2\delta}[E[\sqrt{n}(\frac{T_1+T_2}{R_1+R_2} I_{[R_1\geq 1,\ R_2\geq 1]}-\zeta)]^{2+2\delta}$$

$$+E[\sqrt{n}\,\frac{T_1}{R_1}\, I_{[R_1\geq 1,\ R_2=0]}]^{2+2\delta}+ E[\sqrt{n}\,\frac{T_2}{R_2}\, I_{[R_1=0,\ R_2\geq 1]}]^{2+2\delta}. \tag{5.3.31}$$

Since $T_i \leq n_i(t_i-\eta_i)$ for $i=1,2$, then using (5.3.11) one gets that

$$E[\sqrt{n}\,\frac{T_1}{R_1}\,I_{[R_1\geq 1,\; R_2=0]}]^{2+2\delta} \leq E(\frac{n}{R_1}I_{[R_1\geq 1]}\frac{n_1}{\sqrt{n}}\,(t_1-\eta_1)I_{[R_2=0]})^{2+2\delta}$$

but $E(n/R_1 I_{[R_1\geq 1]})^{2+2\delta} = O(1)$ (5.3.32)

and $E(\frac{n_1}{\sqrt{n}}\,(t_1-\eta_1)I_{[R_2=0]})^{2+2\delta} = (\frac{n_1}{\sqrt{n}}(t_1-\eta_1)^{2+2\delta}\exp[-n_2\zeta(t_2-\eta_2)] \to 0$

as $\min(n_1,n_2) \to \infty$. (5.3.33)

Similarly $E[\sqrt{n}\,\frac{T_2}{R_2}\,I_{[R_1=0,\; R_2\geq 1]}]^{2+2\delta} < \infty$.

Hence it remains to show that the first term on the right hand side of (5.3.31) is bounded. To this effect note that

$$E[\sqrt{n}\,(\frac{T_1+T_2}{R_1+R_2}\,I_{[R_1\geq 1,\; R_2\geq 1]}-\zeta^{-1})^{2+2\delta}$$

$$= E[\frac{n}{R_1+R_2}I_{[R_1\geq 1,\; R_2\geq 1]}(n^{-1/2}(\sum_{j=1}^{n_1} Y_{1j}+\sum_{j=1}^{n_2}Y_{2j}) + n^{-1/2}(n_1(\hat{\eta}_1-\eta_1))$$

$$n^{-1/2}(n_2(\hat{\eta}_2-\eta_2))) + \sqrt{n}\,\zeta^{-1}(I_{[R_1\geq 1,\; R_2\geq 1]}-1))]^{2+2\delta} \qquad (5.3.34)$$

Using the c_δ-inequality, one gets that (5.3.34) is less than

$$4^{1+2\delta}[E(\frac{n}{R_1+R_2})^{2+2\delta}I_{[R_1\geq 1,R_2\geq 1]}[(n^{-1/2}(\sum_{j=1}^{n_1}Y_{1j} + \sum_{j=1}^{n_2}Y_{2j}))^{2+2\delta}$$

$$+ (n^{-1/2}(n_1(\hat{\eta}_1-\eta_1)))^{2+2\delta} + (n^{-1/2}(n_2(\hat{\eta}_2-\eta_2)))^{2+2\delta}]$$

$$+ E(\sqrt{n}\zeta^{-1}(I_{[R_1\geq 1,\; R_2\geq 1]}-1))^{2+2\delta}]. \qquad (5.3.35)$$

We already know that

$$E\left(n_i(\hat{n}_i-n_i)\right)^s = O(1) \text{ for } s > 0 \ i=1,2 \tag{5.3.36}$$

and

$$E\left(\frac{n}{R_1+R_2} I_{[R_1\geq 1,\ R_2\geq 1]}\right)^{4+4\delta} \leq E\left(\frac{n}{n_1}\cdot\frac{n_1}{R_1} I_{[R_1\geq 1]}\right)^{4+4\delta} = O(1) \tag{5.3.37}$$

Also

$$E\left(n^{-1/2}\left(\sum_{j=1}^{n_1} Y_{1j} + \sum_{j=1}^{n_2} Y_{2j}\right)\right)^{4+4\delta} \leq n^{-(2+2\delta)} 2^{3+4\delta}\left[E\left(\left|\sum_{j=1}^{n_1} Y_{1j}\right|\right)^{4+4\delta}\right.$$

$$\left. + E\left(\left|\Sigma_{j=1}^{n_2} Y_{2j}\right|\right)^{4+4\delta}\right] \leq n^{-(2+2\delta)} 2^{3+4\delta}\left(K_1 n_1^{2+2\delta} + K_2 n_2^{2+2\delta}\right) \tag{5.3.38}$$

where in (5.3.38) we have made use of Lemma 3.2.1 in Chapter Three. In addition,

$$E\left(\sqrt{n}\zeta^{-1}\left(I_{[R_1\geq 1,\ R_2\geq 1]}-1\right)\right)^{2+2\delta} \leq \left(\sqrt{n}\zeta^{-1}\right)^{2+2\delta} P(R_1=0) \to 0$$

$$\text{as } \min_{i=1,2} n_i \to \infty, \tag{5.3.39}$$

Hence, using (5.3.36) - (5.3.39) and the Cauchy-Schwarz inequality in (5.3.35), the result follows. □

<u>Remark 3</u> The sufficient statistic in this case, is given by

$$\underset{\sim}{T}_2 = \left(X_{(11)}, X_{(21)}, R_1+R_2, \sum_{j=1}^{R_1} X_{1j} + \sum_{j=1}^{R_2} X_{2j}\right).$$

Again, its density is untractable and may not be complete.

As in the previous case, one can obtain a result analogous to Theorem 2.3.12 by working with all ordered failure times, R_1 and R_2. Hence, one can show that a function $h(\eta_1,\eta_2,\zeta)$ is estimable only if it is of the form $\sum_{r=0}^{\infty} u_r(\eta_1,\eta_2)\zeta^r$ where $u_0(\eta_1,\eta_2)$ does not depend on η_1 and η_2. So neither η_1, η_2 or ζ^{-1} admit an unbiased estimator. In fact even if at least one η_i is known, there still

does not exist an unbiased estimator for ζ^{-1} which is the same result obtained for the with replacement case.

If ζ is known, then $(X_{(11)}, X_{(21)})$ is complete sufficient for (η_1,η_2). In this case the UMVUE of η_i is given by $h(X_{(i1)}) = X_{(i1)} - \zeta^{-1}n_i(1-I_{[X_{(i1)}>t_i]})$ for i=1,2.

Finally, we consider the case when $\eta_1 = \eta_2 = \eta$ and $\zeta_1 = \zeta_2 = \zeta$. Write $Z = \min(X_{(11)}, X_{(21)})$, $R = R_1+R_2$. In this case the MLEs of η and ζ are given respectively by

$$\hat{\eta} = Z(1-I_{[Z>t_1]}) + t_1 I_{[Z>t_1]}; \tag{5.3.40}$$

$$\begin{aligned}\hat{\zeta} = {} & [R/\{\Sigma_{i=1}^{2}\Sigma_{j=1}^{R_2}(X_{(ij)}-\hat{\eta}) + \Sigma_{i=1}^{2}(n_i-R_i)(t_i-\hat{\eta})\}]I_{[R_1>1,R_2>1]} \\ & + [R_1/\{\Sigma_{j=1}^{R_1}(X_{(1j)}-\hat{\eta}) + (n_1-R_1)(t_1-\hat{\eta})\}]I_{[R_1>1,R_2=0]} \\ & +[R_2/\{\Sigma_{j=1}^{R_2}(X_{(2j)}-\hat{\eta}) + (n_2-R_2)(t_2-\hat{\eta})\}]I_{[R_1=0,R_2>1]}\end{aligned} \tag{5.3.41}$$

The modified MLE of η is given by

$$\hat{\hat{\eta}} = \hat{\eta} - (n\hat{\zeta})^{-1}(1-I_{[Z>t_1]}) \tag{5.3.42}$$

Then if we assume (5.3.11), the following results hold true as $n \to \infty$:

(i) $n(\hat{\eta}-\eta) \xrightarrow{L} U$,

where U is exponential with failure rate ζ;

(ii) $E[n(\hat{\eta}-\eta)]^2 \to 2\zeta^{-2}$, $E(n(\hat{\eta}-\eta)) \to \zeta^{-1}$,

(iii) $n(\hat{\hat{\eta}}-\eta) \to U - \zeta^{-1}$,

(iv) · $E[n(\hat{\hat{\eta}}-\eta)]^2 \to \zeta^{-2}$; $E(n(\hat{\hat{\eta}}-\eta)) \to 0$,

(v) $\sqrt{n}(\hat{\zeta}^{-1} - \zeta^{-1}) \xrightarrow{L} N(0,\zeta^{-2}(\lambda p_1+(1-\lambda)p_2)^{-1})$

where $1-p_i = \exp(-\zeta(t_i-\eta))$ $i=1,2$;

(vi) $E[n(\hat{\zeta}^{-1} - \zeta^{-1})^2] \to \zeta^{-2}(\lambda p_1 + (1-\lambda)p_2)^{-1}$.

All proofs are omitted because of the similarity to those given already.

Remark4 As in all previous cases, the derivation of the density of the sufficient statistics $\underset{\sim}{T}_3 = (Z, R_1+R_2, \sum_{j=1}^{R_1} X_{(1j)} + \sum_{j=1}^{R_2} X_{(2j)})$ becomes quite formidable and the induced family of densities may not be complete. Again, by working with the joint density of the ordered failure times for both groups, R_1 and R_2, one can obtain a result analogous to the one obtained for the with replacement case, namely that if a function $h(\eta,\zeta)$ is estimable then it is necessarily of the form $\sum_{r=0}^{\infty} u_r(\eta)\zeta^r$ where $u_0(\eta)$ does not depend on η. Hence, neither η nor ζ^{-1} is estimable. Also, when η is known ζ^{-1} is still not estimable and when ζ is known η has UMVUE given by $h(Z) = Z - ((n_1+n_2)\zeta)^{-1}(1-I_{[Z>t_1]})$.

CHAPTER SIX

GENERALIZED LIKELIHOOD RATIO TESTS FOR THE WITHOUT REPLACEMENT CASE

6.1 Introduction

In this chapter we consider the testing problems (i) - (vii) (see 4.1) when sampling is done without replacement. To be specific, suppose the experiment consists of putting $n_1, n_2, \ldots, n_k$ items to test independently as explained in Section 1.1. Then the likelihood function of all observations is given by (1.1.10) namely,

$$L(\underset{\sim}{\eta}, \underset{\sim}{\zeta}) = \prod_{i \in S}\left[\frac{n_i! \zeta_i^{r_i}}{(n_i - r_i)!} \exp\left[-\zeta_i\left\{\sum_{j=1}^{r_i}(x_{(ij)} - \eta_i) + (n_i - r_i)(t_i - \eta_i)\right\}\right]\right.$$

$$\times I_{[\eta_i < x_{(i1)} < \ldots < x_{(ir_i)} < t_i]} \prod_{j \in \bar{S}}\left[\exp(-n_j \zeta_j (t_j - \eta_j))\right]]$$

The testing problem (i), (ii) and (iii) are considered in Section 6.2. The generalized likelihood ratio test (GLRT) criterion

λ is computed, and the asymptotic distribution of $-2\log\lambda$ is given for both the null and local alternatives. In Section 6.3, the testing problems (iv), (v) and (vi) are considered. In this Section the GLRT criterion λ is computed, and its null distribution is derived. The testing problem (vii) is considered

in Section 6.4. Explicit computation of even the asymptotic null distribution of $-2\log\lambda$ becomes quite formidable in this case, but some conservative test procedure is recommended.

6.2 Testing The Equality of Failure Rates

We shall not make a notational distinction between the rvλ or its value. Before carrying out the actual tests the same preliminary facts given in Chapter 4 are needed. Hence, recall that the likelihood ratio is defined on 2^k distinct regions according to all possible (k-tuple) combinations of $\underset{\sim}{r} = (r_1,\ldots,r_k)$ depending on whether r_i is greater than or equal to zero. Let

$$A_j = \{\underset{\sim}{r}: j \text{ of the } r_i\text{'s equal zero}\},\ j=0,1,\ldots,k. \tag{6.2.1}$$

Hence, for each j, A_j contains $\binom{k}{j}$ elements. Write

$A_j = \bigcup_{\ell=1}^{\binom{k}{j}} B_{j\ell}$, where $\left(B_{j1},\ldots,B_{j\binom{k}{j}}\right)$ constitutes a partition of A_j by elements. Then

$$-2\log\lambda = \Sigma_{j=0}^{k}\Sigma_{\ell=1}^{\binom{k}{j}} (-2\log\lambda)I_{[\underset{\sim}{r}\varepsilon B_{j\ell}]}. \tag{6.2.2}$$

Note that for $j \geq 1$,

$$P\left[(-2\log\lambda)I_{[\underset{\sim}{R}\varepsilon B_{j\ell}]} \neq 0\right]$$

$$\leq P(\text{at least one } R_i=0) \leq \Sigma_{i=1}^{k}P(R_i=0) = \Sigma_{i=1}^{k}\exp[-n_i\zeta_i(t_i-\eta_i)$$

$$\to 0 \text{ as } \min_{1\leq i\leq k} n_i \to \infty. \tag{6.2.3}$$

Also, $P(\underset{\sim}{R} \varepsilon B_{01}) \to 1$ as $\min_{1\leq i\leq k} n_i \to \infty$. Hence,

$$-2\log\lambda = (-2\log\lambda)I_{[\underset{\sim}{R}\varepsilon B_{01}]} + o_p(1), \tag{6.2.4}$$

where by $o_p(1)$, we mean a random variable which converges in probability to zero as $\min(n_1,\ldots,n_k) \to \infty$.

First we consider testing H_{01}. From (1.1.10) it follows that for $\underset{\sim}{R} \varepsilon B_{01}$, MLE of ζ_i is

$$\hat{\zeta}_i = R_i/\{\Sigma_{j=1}^{R_i} X_{(ij)} + (n_i-R_i)t_i - n_i\eta_i\}$$

$$= R_i/\{\Sigma_{j=1}^{n_i}(X_{ij}-\eta_i)I_{[X_{ij}<t_i]} + (t_i-\eta_i)\Sigma_{j=1}^{n_i} I_{[X_{ij}>t_i]}\} \tag{6.2.5}$$

$i=1,\ldots,k$. Also, under H_{01}, for $\underset{\sim}{R} \varepsilon B_{01}$, MLE of the common failure rate ζ is

$$\hat{\zeta} = R/\{\Sigma_{i=1}^{k}\Sigma_{j=1}^{n_i}(X_{ij}-\eta_i)I_{[X_{ij}<t_i]} + \Sigma_{i=1}^{k}(t_i-\eta_i)\Sigma_{j=1}^{n_i} I_{[X_{ij}>t_i]}\}$$

where $R = \Sigma_{i=1}^{k} R_i$ (6.2.6)

Let $T_i = \Sigma_{i=1}^{n_i}(X_{ij}-\eta_i)I_{[X_{ij}<t_i]} + (t_i-\eta_i)\Sigma_{j=1}^{n_i} I_{[X_{ij}>t_i]}$

$i=1,\ldots,k$, and $T = \Sigma_{i=1}^{k} T_i$. Then, one can write

$$\hat{\zeta}_i = R_i/T_i \ (i=1,\ldots,k) \text{ and } \hat{\zeta} = R/T, \tag{6.2.7}$$

when $\underset{\sim}{R} \varepsilon B_{01}$. Substituting these estimators $\hat{\zeta}_i$'s and $\hat{\zeta}$ for ζ_i's and ζ in the likelihood function given in (1.1.10), for $\underset{\sim}{R} \varepsilon B_{01}$, the GLRT criterion λ is given by

$$\lambda = \{R^R/(\Pi_{i=1}^{k} R_i^{R_i})\}\ (\Pi_{i=1}^{k} T_i^{R_i}/T^R). \tag{6.2.8}$$

Using the strong law of large numbers (SLLN), as $n_i \to \infty$,

$T_i/n_i \xrightarrow{a.s.} E[(X_{i1}-\eta_i)I_{[X_{i1}<t_i]} + (t_i-\eta_i)I_{[X_{i1}>t_i]}] = p_i\zeta_i^{-1}$. In the above, we have used the fact that $X_{ij}-\eta_i$'s are iid exponential with failure rate ζ_i^{-1} and $1-p_i = \exp(-\zeta_i(t_i-\eta_i))$. Next observe that

$$n_i^{-1/2}(R_i-\zeta_iT_i) = n_i^{-1/2}\Sigma_{j=1}^{n_i}Z_{ij}, \tag{6.2.9}$$

where

$$Z_{ij} = I_{[X_{ij}<t_i]}-\zeta_i(X_{ij}-\eta_i)I_{[X_{ij}<t_i]}-\zeta_i(t_i-\eta_i)I_{[X_{ij}>t_i]}, \tag{6.2.10}$$

for $i=1,2,\ldots k$.

Note that $Z_{i1},\ldots,Z_{in_i}$ are iid with $E(Z_{i1}) = 0$ and $V(Z_{i1}) = p_i$. Hence, using the central limit theorem, as $n_i \to \infty$,

$$n_i^{-1/2}(R_i-\zeta_iT_i) \xrightarrow{L} N(0,p_i). \tag{6.2.11}$$

Since $T_i/n_i \xrightarrow{a.s.} p_i\zeta_i^{-1}$ as $n_i \to \infty$, if follows form (6.2.11) that

$$T_i^{-1/2}(R_i-\zeta_iT_i) \xrightarrow{L} N(0,\zeta_i). \tag{6.2.12}$$

In order to find the limiting distiribution of $-2\log\lambda$, we make the following assumption

$$\lim_{n\to\infty} n_i/n = \lambda_i,\ 0<\lambda_i<1 \text{ and } \Sigma_{i=1}^k\lambda_i = 1 \text{ where } n = \Sigma_{i=1}^k n_i. \tag{6.2.13}$$

We now prove the first theorem of this section which provides the asymptotic null distribution of $(-2\log\lambda)I_{[\underset{\sim}{R}\,\varepsilon\, B_{01}]}$. Since,

$$P(\underset{\sim}{R}\notin B_{01}) < \Sigma_{i=1}^k P(R_i=0) = \Sigma_{i=1}^k\exp(-n_i\zeta_i(t_i-\eta_i)) \to 0 \text{ as } \min_{1<i<k} n_i \to \infty$$

which holds under (6.2.13)), it follows that under (6.2.13),

$-2\log\lambda$ has the same limiting distribution as $(-2\log\lambda)I_{[\underset{\sim}{R}\varepsilon B_{01}]}$

<u>Theorem 6.2.1</u> <u>If</u> (6.2.13) <u>holds</u>, <u>then as</u> $n \to \infty$,

<u>under</u> $H_{01}: \zeta_1 = \dots = \zeta_k = \zeta$, $-2\log\lambda \xrightarrow{L} \chi^2_{k-1}$.

<u>Proof</u> Since, $\zeta_1 = \dots = \zeta_k = \zeta$, using Taylor expansion for $\underset{\sim}{R} \varepsilon B_{01}$, one gets from (6.2.8),

$$-2\log\lambda = 2[\Sigma_{i=1}^{k} R_i \log(R_i/T_i) - R\log(R/T)]$$

$$= 2[\Sigma_{i=1}^{R} R_i \log(R_i/(\zeta T_i)) - R\log(R/(\zeta T))]$$

$$= 2[\Sigma_{i=1}^{k} (R_i-\zeta T_i+\zeta T_i)\log(1+(R_i-\zeta T_i)(\zeta T_i)^{-1})$$

$$-R\log(1+(R-\zeta T)\zeta T)^{-1})$$

$$= 2[\Sigma_{i=1}^{k} (R_i-\zeta T_i+\zeta T_i)\{\frac{R_i-\zeta T_i}{\zeta T_i} - \frac{(R_i-\zeta T_i)^2}{2(\zeta T_i)^2} + \frac{(R_i-\zeta T_i)^3}{6(\zeta T_i)^3(1+\phi_i\frac{R_i-\zeta T_i}{\zeta T_i})^3}\}$$

$$-(R-\zeta T+\zeta T)\ \{\frac{R-\zeta T}{\zeta T} - \frac{(R-\zeta T)^2}{2(\zeta T)^2} + \frac{(R-\zeta T)^3}{6(\zeta T)^3(1+\phi\frac{R-\zeta T}{\zeta T})^3}\}] \quad (6.2.14)$$

where $0 < \phi_i < 1$ $(i=1,\dots,k)$ and $0 < \phi < 1$. Using (6.2.12), (6.2.13) and the fact that $T_i/n_i \xrightarrow{a.s.} p_i\zeta^{-1}$ as $n_i \to \infty$, it follows from (6.2.14) after some simplification that

$$(-2\log\lambda)I_{[\underset{\sim}{R}\varepsilon B_{01}]}$$

$$= \{\Sigma_{i=1}^{k}(R_i-\zeta T_i)^2(\zeta T_i)^{-1} - (R-\zeta T)^2(\zeta T)^{-1}\}I_{[\underset{\sim}{R}\varepsilon B_{01}]} + o_p(1) \quad (6.2.15)$$

Hence, for proving the theorem, it suffices to show that

$$Q = \{\Sigma_{i=1}^{k}(R_i-\zeta T_i)^2(\zeta T_i)^{-1} - (R-\zeta T)^2(\zeta T)^{-1}\}I_{[\underset{\sim}{R}\varepsilon B_{01}]}$$

$$\xrightarrow{L} \chi^2_{k-1} \text{ under } H_{01}. \tag{6.2.16}$$

To prove (6.2.16), write $Q = (\underset{\sim}{Y}'\underset{\sim}{A}\underset{\sim}{Y})I_{[\underset{\sim}{R}\varepsilon B_{01}]}$

where $\underset{\sim}{Y} = (Y_1,\ldots,Y_k)'$ with $Y_i = (R_i-\zeta T_i)(\zeta T_i)^{-1/2}$ $(i=1,\ldots,k)$,

and $\underset{\sim}{A} = \underset{\sim}{I}_k - \underset{\sim}{u}\underset{\sim}{u}'$, $\underset{\sim}{u} = ((T_1/T)^{1/2},\ldots,(T_k/T)^{1/2})'$.

Since $T_i/n_i \xrightarrow{a.s.} p_i\zeta^{-1}$ as $n \to \infty$ and (6.2.13), holds , it follows

that $\underset{\sim}{A} \xrightarrow{a.s.} I_k - \underset{\sim}{d}\underset{\sim}{d}'$, where $\underset{\sim}{d}' = (d_1,\ldots,d_k)$ with

$d_i = (p_i\lambda_i/\Sigma_{i=1}^{k}p_i\lambda_i)^{1/2}$ and $p_i = 1-\exp[-\zeta(t_i-\eta_i)]$.

Next observe that $\underset{\sim}{I}_k - \underset{\sim}{d}\underset{\sim}{d}'$ is symmetric, idempotent with rank $(\underset{\sim}{I}_k-\underset{\sim}{d}\underset{\sim}{d}') = \operatorname{tr}(\underset{\sim}{I}_k-\underset{\sim}{d}\underset{\sim}{d}') = k-1$.

Moreover $\underset{\sim}{Y} \xrightarrow{L} N_k(\underset{\sim}{0},\underset{\sim}{I}_k)$ as $n \to \infty$. Hence, using Lemma 4.2.1 and Slutsky's Theorem, $\underset{\sim}{Y}'\underset{\sim}{A}\underset{\sim}{Y} \xrightarrow{L} \chi^2_{k-1}$ under H_{01}.

Since $I_{[\underset{\sim}{R}\varepsilon B_{01}]} \xrightarrow{P} 1$ as $n \to \infty$, one gets (6.2.16). □

Next consider the sequence of local alternative $\zeta_i = \zeta + \Delta_i n_i^{-1/2}$ $(i=1,2,\ldots k)$. Use the same Taylor expansion for $-2\log\lambda$ as in (6.2.14). Now write

$$Y_i = (R_i-\zeta T_i)(\zeta T_i)^{-1/2} = (R_i-\zeta_i T_i)(\zeta T_i)^{-1/2} + (\zeta_i-\zeta)T_i(\zeta T_i)^{-1/2}$$

$$= (\zeta_i/\zeta)^{1/2}(R_i-\zeta_i T_i)(\zeta_i T_i)^{-1/2} + \Delta_i(T_i/n_i)^{1/2}\zeta^{-1/2}.$$

Note that in view of (6.2.13), $\zeta_i/\zeta \to 1$ as $n \to \infty$.

Also, $T_i/n_i \xrightarrow{a.s.} p_i\zeta^{-1}$ as $n \to \infty$, since $\zeta_i^{-1} \to \zeta^{-1}$ as $n \to \infty$.

Here $p_i = 1 - \exp[-\zeta(t_i - \eta_i)]$. Hence, using independence of the Y_i's, $\underset{\sim}{Y} \xrightarrow{L} N_k(\underset{\sim}{\delta}, I_k)$, where $\underset{\sim}{\delta}' = (\delta_1, \ldots \delta_k)$, $\delta_i = \Delta_i p_i^{+1/2} \zeta^{-1}$.

As before $\underset{\sim}{A} \xrightarrow{a.s.} I_k - \underset{\sim}{dd}'$ where $\underset{\sim}{d} = (d_1, \ldots, d_k)$ with $d_i = \{\lambda_i p_i / \Sigma_{i=1}^k \lambda_i p_i\}^{1/2}$. Hence, using Lemma 4.2.1 $\underset{\sim}{Y}' A \underset{\sim}{Y} \xrightarrow{L} \chi^2_{k-1}(\tau_1)$

where $\tau_1 = \underset{\sim}{\delta}'(\underset{\sim}{I}_k - \underset{\sim}{dd}')\underset{\sim}{\delta} = \Sigma_{i=1}^k \delta_i^2 - (\Sigma_{i=1}^k \delta_i d_i)^2$.

Next we consider testing H_{02}. In this case, the MLE of η is $\hat{\eta} = \min_{1 \le i \le k} X_{(i1)}$, and the MLE of ζ_i is

$$\hat{\zeta}_i = R_i / \{\Sigma_{j=1}^{R_i} X_{(ij)} + (n_i - R_i)t_i - n_i \hat{\eta}\}$$

$$= R_i / \{\Sigma_{j=1}^{n_i} (X_{ij} - \eta) I_{[X_{ij} < t_i]} + (t_i - \eta)\Sigma_{j=1}^{n_i} I_{[X_{ij} > t_i]} - n_i(\hat{\eta} - \eta)\}$$

$$= R_i / \{T_i - n_i(\hat{\eta} - \eta)\} = R_i / T_i^* \text{ (say)} \qquad (6.2.17)$$

Under H_{02}, $\zeta_1 = \cdots = \zeta_k = \zeta$, and the MLE of ζ is

$$\hat{\zeta} = R/\{T - n(\hat{\eta} - \eta)\} = R/T^* \text{(say)} \qquad (6.2.18)$$

In this case, for $\underset{\sim}{R} \in B_{01}$, the GLRT criterion λ, reduces to

$$\lambda = \prod_{i=\lambda}^{k} (T_i^*/R_i)^{R_i} (R/T^*)^R \qquad (6.2.19)$$

Next, observe that if (6.2.13) holds, $T_i^*/n_i \xrightarrow{a.s.} p_i \zeta_i^{-1}$ as $n \to \infty$, since $\hat{\eta} \xrightarrow{a.s.} \eta$ as $n \to \infty$. Moreover, if (6.2.13) holds, since $n(\hat{\eta} - \eta) = 0_p(1)$, $n^{-1/2}(\hat{\eta} - \eta) \xrightarrow{P} 0$. Accordingly, under H_{02}, $Z_i = (\zeta T_i^*)^{-1/2}(R_i - \zeta T_i^*)$

Moreover, if (6.2.13) holds, since $n(\hat{\eta}-\eta) = O_p(1)$, $n^{-1/2}(\hat{\eta}-\eta) \xrightarrow{P} 0$. Accordingly, under H_{02}, $Z_i = (\zeta T_i^*)^{-1/2}(R_i-\zeta T_i^*)$

$$= (\zeta T_i^*)^{-1/2}[(R_i-\zeta T_i) + \zeta(T_i-T_i^*)]$$

$$= (n_i/\zeta T_i^*)^{1/2} n_i^{-1/2}[(R_i-\zeta T_i) + n_i\zeta(\hat{\eta}-\eta)]$$

$$\xrightarrow{L} N(0,1) \tag{6.2.20}$$

Note that the Z_i's are not independent since they all share the same $\hat{\eta}$.

Following a Taylor expansion similar to (6.2.14), one gets as in (6.2.15)

$$(-2\log\lambda)I_{[\underset{\sim}{R}\varepsilon B_{01}]} = Q_o I_{[\underset{\sim}{R}_i\varepsilon B_{01}]} + o_p(1) \tag{6.2.21}$$

where $Q_o = \underset{\sim}{Z}'\underset{\sim}{A}_o\underset{\sim}{Z}$ with $\underset{\sim}{Z}' = (Z_1,\ldots,Z_k)$, $\underset{\sim}{A}_o = \underset{\sim}{I}_k - \underset{\sim}{\mu}_o\underset{\sim}{\mu}_o'$,

$\underset{\sim}{\mu}_o' = ((T_1^*/T^*)^{1/2},\ldots,(T_k^*/T^*)^{1/2})$. Also

$\underset{\sim}{A}_o \xrightarrow{a.s.} I_k - \underset{\sim}{d}\underset{\sim}{d}'$ as $n \to \infty$ when (6.2.13) holds. Write

$$W_i = (n_i/\zeta T_i)^{1/2} n_i^{-1/2}(R_i-\zeta T_i), \quad i=1,2,\ldots,k,$$

where $T_i = \Sigma_{j=1}^{n_i}\left[(X_{ij}-\eta)I_{[X_{ij}<t_i]} + (t_i-\eta)I_{[X_{ij}>t_i]}\right]$
and η is unknown.

Note $W_i \xrightarrow{L} N(0,1)$ and the W_i's are independently distributed. Also $Z_i - W_i \xrightarrow{P} 0$ $(i=1,2,\ldots k)$ as $\min_{i=1,\ldots,k} n_i \to \infty$.

Using the Cramer-Wold device, Slutsky's theorem, Lemma 4.2.1 and the fact that $P(\underset{\sim}{R}\varepsilon B_{01}) \to 1$ as $n \to \infty$, one gets

$\zeta_i = \zeta + \Delta_i n_i^{-1/2}$ since from the first line of (6.2.20)

$$Z_i = (\zeta T_i^*)^{-1/2}[(R_i-\zeta T_i^*) + (\zeta_i-\zeta)T_i^*]$$

$$=(\zeta_i/\zeta)^{1/2}(\zeta_i T_i^*)^{-1/2}(R_i-\zeta_i T_i^*) + \Delta_i(T_i^*/n_i)^{1/2}\zeta^{-1/2}$$

$$\xrightarrow{L} N(\delta_i^*,1) \qquad (6.2.22)$$

where $\delta_i^* = \Delta_i p_i^{*+1/2}\zeta^{-1}$, and $p_i^* = 1-\exp[-\zeta(t_i-\eta)]$ and η is unknown.

Since the Z_i's are not independent, we write

$$W_i = (n_i/\zeta T_i)^{1/2} n^{-1/2}(R_i-\zeta_i T_i) + \Delta_i(T_i/n_i)^{1/2}\zeta^{-1/2}$$

$$= \left(\frac{\zeta_i}{\zeta}\right)^{1/2}(\zeta_i T_i)^{-1/2}(R_i-\zeta_i T_i) + \Delta_i(T_i/n_i)^{1/2}\zeta^{-1/2}$$

and observe that $W_i \xrightarrow{L} N(\delta_i^*,1)$ and $W_i-Z_i \xrightarrow{P} 0$. Since, the W_i's are independent, $\underset{\sim}{W} \xrightarrow{L} N(\underset{\sim}{\delta}^*, I_k)$ where $\underset{\sim}{W}' = (W_1,\ldots,W_k)$ and $\underset{\sim}{\delta}^* = (\delta_1^*,\ldots,\delta_k^*)$.

Arguing as before for the given sequence of local alternatives,

$-2\log\lambda \xrightarrow{L} \chi^2_{k-1}(\tau_2)$ where $\tau_2 = \Sigma_{i=1}^k(\delta_i^*)^2 - (\Sigma_{i=1}^k \delta_i^* d_i^*)^2$.

Here $d_i^* = \{\lambda_i p_i^*/\Sigma_{i=1}^k \lambda_i p_i^*\}^{1/2}$.

Finally, in this section, we consider testing H_{03}. In this case, the MLE of η_i is $\hat{\eta}_i = X_{(i1)}$ and for $\underset{\sim}{R}\varepsilon B_{01}$, ζ_i has MLE $\hat{\zeta}_i = R_i/T_{i0}$, where

$$T_{i0} = \Sigma_{j=1}^{n_i} X_{ij} I_{[X_{ij}<t_i]} + t_i \Sigma_{j=1}^{n_i} I_{[X_{ij}>t_i]} - n_i\hat{\eta}_i$$

$$= \Sigma_{j=1}^{n_i}(X_{ij}-\eta_i)I_{[X_{ij}<t_i]}+(t_i-\eta_i)\Sigma_{j=1}^{n_i}I_{[X_{ij}>t_i]}$$

$$-n_i(\hat{\eta}_i-\eta_i) \tag{6.2.23}$$

Under H_{03}, the MLE of ζ is $\hat{\zeta} = R/T_0$ where $T_0 = \Sigma_{i=1}^{R}T_{i0}$. Essentially, repetition of the previous steps now give that under H_{03}, $-2\log\lambda \xrightarrow{L} \chi^2_{k-1}$, and under the sequence of local alternatives $\zeta_i = \zeta + \Delta_i n_i^{-1/2}$ $(i=1,\ldots,k)$, $-2\log\lambda \xrightarrow{L} \chi^2_{k-1}(\tau_3)$ where $\tau_3 = (\Sigma_{i=1}^{k}\delta_i^{**})^2 - (\Sigma_{i=1}^{k}\delta_i^{**}d_i^{**})^2$, with $\delta_i^{**} = \Delta_i p_i^{**\,1/2}\zeta^{-1}$, $p_i^{**} = 1-\exp[-\zeta(t_i-\eta_i)]$ (here the η_i's are all unknown) and $d_i^{**} = (\lambda_i p_i^{**}/\Sigma_{i=1}^{k}\lambda_i p_i^{**})^{1/2}$

6.3 Testing The Equality of Locations

First consider testing H_{04}. Note that $\underset{\sim}{R} \in B_{01} \iff X_{(i1)} < t_i$ for all $i = 1,\ldots,k$. Accordingly for $\underset{\sim}{R} \in B_{01}$, the MLE of η_i is $\hat{\eta}_i = X_{(i1)}$, and under H_{04}, the MLE of the common location parameter η is $\hat{\eta} = \min_{1<i<k} X_{(i1)}$. As in Chapter Four, the GLRT criterion λ is given by

$$\lambda = \exp[-\Sigma_{i=1}^{k} n_i(\hat{\eta}_i-\hat{\eta})], \tag{6.3.1}$$

when $\underset{\sim}{R}\in B_{01}$. Since the $X_{(i1)}$'s have the same distribution as in the with replacement case, using the same argument as given in Theorem 4.3.2 of Chapter Four, one gets

$$-2\log\lambda \xrightarrow{L} \chi^2_{k-1} \text{ under } H_{04} \text{ as } n \to \infty \tag{6.3.2}$$

provided (6.2.13) holds.

Next we consider testing H_{05}. Let $\underset{\sim}{R}\varepsilon B_{01}$. In this case, the MLE of η_i is $\hat{\eta}_i = X_{(i1)}$, and from Section 6.2, the MLE of the common scale parameter ζ is $\hat{\zeta} = R/T_0$. Under H_{05}, the MLE of the common location parameters η is $\hat{\eta} = \min_{1\le i\le k} X_{(i1)}$, while the MLE of ζ is $\hat{\hat{\zeta}} = R/T^*$. Accordingly for $\underset{\sim}{R}\varepsilon B_{01}$, the GLRT criterion λ is given by

$$\lambda = (\hat{\zeta}/\hat{\hat{\zeta}})^R = (T_0/T^*)^R \tag{6.3.3}$$

Hence,

$$-2\log\lambda = -2R\log\left(1 - \frac{T^*-T_0}{T^*}\right). \tag{6.3.4}$$

From the previous section,

$$\begin{aligned} T^*-T_0 &= \{T-n(\hat{\eta}-\eta)\} - \{T - \Sigma_{i=1}^k n_i(\hat{\eta}_i-\eta)\} \\ &= \Sigma_{i=1}^k n_i(\hat{\eta}_i-\eta) > 0 \end{aligned}$$

with probability 1 so that $0 < (T^*-T_0)/T^* < 1$ with probability 1. Now using the inequality

$$-x - \frac{x^2}{2(1-x)} < \log(1-x) < -x \text{ for } 0 < x < 1,$$

it follows from (6.3.4) that with probability 1,

$$\begin{aligned} & 2R(T^*-T_0)/T^* - \frac{R(T^*-T_0)^2}{T^*T_0} \\ & > -2\log\lambda \\ & > 2R(T^*-T_0)/T^* \end{aligned} \tag{6.3.5}$$

We have seen already that under H_{05}: $\eta_1 = \dots = \eta_k = \eta$, $2\zeta(T^*-T_0)$ $= 2\zeta\ \Sigma_{i=1}^k\ n_i(\hat{\eta}_i-\hat{\eta}) \xrightarrow{L} \chi^2_{2(k-1)}$.

Also, $R/(\zeta T^*) \xrightarrow{a.s.} 1$. Moreover, since $T^*-T_0 = O_p(1)$, $R/T^* \xrightarrow{a.s.} \zeta$ and

$T_0 \xrightarrow{a.s.} +\infty$ as $n \to \infty$, $R(T^*-T_0)^2/(T^*T_0) \xrightarrow{P} 0$ as $n \to \infty$.

Hence, it follows from (6.3.5) that $-2\log\lambda \xrightarrow{L} \chi^2_{2(k-1)}$ under H_{05}.

Finally, we consider testing H_{06}. For $\underset{\sim}{R}\varepsilon B_{01}$, the MLE of η_i is $\hat{\eta}_i = X_{(i1)}$, while the MLE of ζ_i is $\hat{\zeta}_i = R_i/T_{i0}$. When H_{06} holds, the MLE of the common location parameter η is $\hat{\eta}$ $= \min\limits_{1\leq i\leq k} X_{(i1)}$ and, the MLE of ζ_i is $\hat{\hat{\zeta}}_i = R_i/T_i^*$. Then the GLRT criterion λ is

$$\lambda = \prod_{i=1}^{k} (\hat{\hat{\zeta}}_i/\hat{\zeta}_i)^{R_i}. \tag{6.3.6}$$

Hence, for $\underset{\sim}{R}\varepsilon B_{01}$,

$$\begin{aligned} &-2\log\lambda \\ &= -2\Sigma_{i=1}^{k} R_i \log(\hat{\hat{\zeta}}_i/\hat{\zeta}_i) \\ &= -2\Sigma_{i=1}^{k} R_i \log(T_{i0}/T_i^*) \\ &= -2\Sigma_{i=1}^{k} R_i \log\left(1 - \frac{T_i^*-T_{i0}}{T_i^*}\right). \end{aligned} \tag{6.3.7}$$

Since, $T_i^* - T_{i0} = n_i(\hat{\eta}_i - \hat{\eta}) > 0$ with probability 1, using an inequality similar to (6.3.5), it follows from (6.3.7) that

$$\begin{aligned} &2\Sigma_{i=1}^{k} R_i\{(T_i^*-T_{io})/T_i^* - (T_i^*-T_{i0})^2/(T_i^*T_{i0})\} \\ &\geq -2\log\lambda \\ &\geq 2\Sigma_{i=1}^{k} R_i(T_i^*-T_{i0})/T_i^* . \end{aligned} \tag{6.3.8}$$

When H_{06} holds, $R_i/(\zeta_i T_i^*) \xrightarrow{a.s.} 1$, and

$$2\Sigma_{i=1}^{k} \zeta_i(T_i^*-T_{i0}) = 2\Sigma_{i=1}^{k} n_i\zeta_i(\hat{\eta}_i-\hat{\eta}) \xrightarrow{L} \chi^2_{2(k-1)} \tag{6.3.9}$$

The proof of this result is similar to the proof of Theorem (4.3.2) of Chapter Four. Next we write

$$2\Sigma_{i=1}^{k} R_i(T_i^*-T_{i0})/T_i^* = \Sigma_{i=1}^{k} 2((R_i/T_i^*)-\zeta_i)(T_i^*-T_{i0})$$

$$+ \Sigma_{i=1}^{k} 2\zeta_i (T_i^* - T_{i0}) \tag{6.3.10}$$

and note that under H_{06},

$$\Sigma_{i=1}^{k} -2((R_i/T_i^*)-\zeta_i)n_i(\hat{\eta}-\eta) < \Sigma_{i=1}^{k} 2((R_i/T_i^*)-\zeta_i)n_i(\hat{\eta}_i-\hat{\eta})$$

$$< \Sigma_{i=1}^{k} 2((R_i/T_i^*)-\zeta_i)n_i(\hat{\eta}_i-\eta). \tag{6.3.11}$$

Recall that $n_i\zeta_i(\hat{\eta}_i-\eta) \xrightarrow{L} U$, where U is distributed as an exponential random variable with mean equal to one.

Also $\Sigma_{i=1}^{k} n_i\zeta_i(\hat{\eta}-\eta) \xrightarrow{L} U$ and $(R_i/T_i^*)- \zeta_i \xrightarrow{P} 0$ as $\min\limits_{1<i<k} n_i \to \infty$.

Hence both upper and lower bounds in (6.3.11) go to zero a.s. which implies that

$$\Sigma_{i=1}^{k} 2((R_i/T_i^*)- \zeta_i)n_i(\hat{\eta}_i-\hat{\eta}) \xrightarrow{P} 0 \text{ as } \min_{1<i<k} n_i \to \infty. \tag{6.3.12}$$

Combining (6.3.9) - (6.3.12) it now follows that

$$2\Sigma_{i=1}^{k} R_i(T_i^*-T_{i0})/T_i^* \xrightarrow{L} \chi^2_{2(k-1)} \tag{6.3.13}$$

Also, as $\min\limits_{1<i<k} n_i \to \infty$,

$$2\Sigma_{i=1}^{k} R_i(T_i-T_{i0})^2/T_i^*T_{i0} \xrightarrow{P} 0 \tag{6.3.14}$$

since

$$(T_i-T_{i0})^2 = (n_i(\hat{\eta}_i-\hat{\eta}))^2 = 0_p(1)$$

and $R_i/T_i^* \xrightarrow{P} \zeta_i$ while $1/T_{i0} \xrightarrow{a.s.} 0$.

Hence, using (6.3.8) and from (6.3.13) and (6.3.14) it now follows that under H_{06},

$$-2\log\lambda \xrightarrow{L} \chi^2_{2(k-1)}.$$

6.4 Testing For Location and Scale Parameters

In this section we test H_{07}. Note that for $\underset{\sim}{R}\epsilon B_{01}$, the MLE of η_i is $\hat{\eta}_i = X_{(i1)}$, while the MLE of ζ_i is $\hat{\zeta}_i = R_i/T_{i0}$. Under H_{07}

for $\underset{\sim}{R}\varepsilon B_{01}$, the MLE of the common location parameter η is

$\hat{\eta} = \min_{1 \leq i \leq k} X_{(i1)}$, while the MLE of the common scale parameter is

$\hat{\zeta} = R/T^*$.

Accordingly, for $\underset{\sim}{R}\varepsilon B_{01}$, the GLRT criterion is given by

$$\lambda = \prod_{i=1}^{k} (\hat{\zeta}/\hat{\zeta}_i)^{R_i} = \prod_{i=1}^{k} \left(\frac{R}{T^*} \cdot \frac{T_{i0}}{R_i}\right)^{R_i}$$

Recall that $T^* = T - n(\hat{\eta}-\eta)$

$$= \Sigma_{i=1}^{k}(T_i - n_i(\hat{\eta}-\eta)) = \Sigma_{i=1}^{k} T_i^* \text{ (say)}$$

where

$$T_i = \Sigma_{i=1}^{n_i}\{(X_{(ij)}-\eta)I_{[X_{(ij)}<t_i]} + (t_i-\eta)I_{[X_{(ij)}>t_i]}\} \quad i=1,2,\ldots,k$$

and

$$T_{i0} = \Sigma_{j=1}^{k}\{(X_{ij}-\eta_i)I_{[X_{ij}<t_i]} + (t_i-\eta_i)I_{[X_{ij}>t_i]}\} - n_i(\hat{\eta}_i-\eta_i)$$

for $i=1,2,\ldots,k$.

Hence for $\underset{\sim}{R}\varepsilon B_{01}$,

$$-2\log\lambda = 2(\Sigma_{i=1}^{k} R_i \log\hat{\zeta}_i - R\log\hat{\zeta})$$

$$= 2[\Sigma_{i=1}^{k} R_i \log(\hat{\zeta}_i/\zeta) - R \log (\hat{\zeta}/\zeta)]$$

$$= 2[\Sigma_{i=1}^{k} R_i \log (R_i T_i^*/\zeta T_i^* T_{i0}) - R \log (R/T^*\zeta)$$

$$= 2[\Sigma_{i=1}^{k} R_i \log(R_i/\zeta T_i^*) - \Sigma_{i=1}^{k} R_i \log(T_{i0}/T_i^*) - R\log(R/T^*\zeta)]$$

$$= 2[\Sigma_{i=1}^{k}(R_i-\zeta T_i + \zeta T_i)\log(1 + (R_i-\zeta T_i)(\zeta T_i)^{-1})$$

$$- (R - \zeta T + \zeta T)\log(1 + (R - \zeta T)(\zeta T)^{-1})]$$

$$- 2\Sigma_{i=1}^{k} R_i \log\left(1 - \frac{T_i^* - T_{i0}}{T_i^*}\right) \tag{6.4.1}$$

Now, combine the arguments used for testing H_{01} and H_{02} as well as H_{06}. This leads to

$$-2\log\lambda I_{[\underset{\sim}{R}\varepsilon B_{01}]} = (Q_1+Q_2)I_{[\underset{\sim}{R}\varepsilon B_{01}]} + o_p(1) \tag{6.4.2}$$

as $n \to \infty$, where

$$Q_1 = \Sigma_{i=1}^{k}(R_i - \zeta T_i^*)^2(\zeta T_i^*)^{-1} - (R-\zeta T^*)^2(\zeta T^*)^{-1} \tag{6.4.3}$$

$$Q_2 = 2\Sigma_{i=1}^{k} R_i(T_i^* - T_{i0})/T_i^* \tag{6.4.4}$$

Under H_{07}, for $\underset{\sim}{R}\varepsilon B_{01}$, $Q_1 \xrightarrow{L} X^2_{(k-1)}$ and $Q_2 \xrightarrow{L} X^2_{2(k-1)}$. Also $I_{[\underset{\sim}{R}\varepsilon B_{01}]} \xrightarrow{a.s.} 1$

However Q_1 and Q_2 are <u>not</u> independent. Thus, under H_{07}, $-2\log\lambda \xrightarrow{L} Y_1+Y_2$ where $Y_1 \sim X^2_{k-1}$ and $Y_2 \sim X^2_{2(k-1)}$, but Y_1 and Y_2 are not necessarily independent.Hence, as explained in H_{07} for the with replacement case, if we reject when

$-2\log\lambda > K_1+K_2$ where $K_1 = X^2_{k-1;\alpha/2}$ and $K_2=X^2_{2(k-1);\alpha/2}$

and $X^2_{n;\alpha}$ denotes the upper $100\alpha\%$ of X^2_n, then it follows that asymptotically the proposed test procedure has size less than or equal to α.

CHAPTER SEVEN

FUTURE RESEARCH

In this investigation we have considered inference regarding the parameters of one or more location and scale parameter exponentials under Type I censoring for both the cases when sampling is done with replacement and without replacement.

Future research related to the above area can proceed along several lines. First, we have only considered singly Type I censored data. More general censoring mechanisms such as multiple Type I censoring with random or non-random censoring times as well as hybrid censoring, which is a combination of Types I and II censoring, are of interest from a theoretical as well as practical point of view.

Secondly, we have restricted ourselves to the two parameter exponential family. Other parametric families could be studied under the same setting as covered by the present investigation or under more general setting as described in the previous paragraph.

In particular, the Weibull family of densities which occupies an important position in life data analysis would be an interesting candidate.

Moreover, one can study the properties of estimators and test procedures obtained under other modes of sampling such as

sequential sampling of censored data. It is also of importance to note that the methods presented in this manuscript for hypothesis testing in a multisample setting as well as the properties of the MLEs or the modified MLEs in all the cases are more applicable in large sample situations. The adequacy of such methods in small samples remains to be assessed and methods for small samples need to be investigated.

BIBLIOGRAPHY

Bain, L.J.(1978). Statistical Analysis of Reliability and Life-Testing Models. Marcel-Decker Inc., New York, NY.

Barlow, R.E. and Proschan, F. (1981). Statistical Theory of Reliability and Life Testing, 2nd Edition. McArdle Press, Inc., Baltimore, MD.

Bartholomew, D.J. (1963). The sampling distribution of an estimate arising in life testing. Technometrics, 5, (3), 361-374.

Butler, J.P., and Rees, D.A. (1974). Exploration of statistical fatigue failure characteristics of 0.063-inch mill-annealed Ti6Al-4V sheet and 0.050-inch heat-treated 17-7 Ph steel sheet under simulated flight-by-flight loading. Technical report AFML-TR-74-269, Air Force Materials Laboratory, Air Force Systems Command, Wright-Patterson AFB, OH.

Chiou, W.J. and Cohen, A. (1984). Estimating the common location parameter of exponential distributions with censored samples. Naval Research Logistic Quarterly, 31, 475-482.

David, H.A. (1981). Order Statistics, 2nd Edition. Wiley, New York, NY.

Deemer Jr., W.L. and Votaw Jr., D.F. (1955). Estimation of parameters of truncated or censored exponential distributions. Ann. Math. Statist., 21, 498-504.

Engelhardt, M., and Bain, L.J. (1975). Tests of two-parameter exponentiality against three-parameter weibull alternatives. Technometrics, 17, 353-356.

Epstein, B. and Tsao, C.K. (1953). Some tests based on ordered observations from two exponential populations. Ann. Math. Statist., 24, 458-466.

Ghosh, M. and Razmpour A. (1984). Estimation of the common location parameter of several exponentials. Sankhya' A, 46, 383-394.

Hogg, R.V. (1956). On the distribution of the likelihood ratio. Ann. Math Statist., 27, 529-532.

Khatri, C.G. (1974). On testing the equality of location parameters in k-censored exponential distributions. Australian J. Statist., 16, 1-10.

Lawless, J.F. (1982). Statistical Models and Methods for Lifetime Data. Wiley, New York, NY.

Mann, N.R., Schafer, R.E. and Singpurwalla, N.D. (1974). Methods for Statistical Analysis of Reliability and Life Data. Wiley, New York, NY.

Nelson, W. (1982). Applied Life Data Analysis. Wiley, New York, NY.

Serfling, R.J. (1980). Approximation Theorems of Mathematical Statistics. Wiley, New York, NY.

Sukhatme, P.V. (1937). Tests of significance for samples of the x^2 population with two degrees of freedom. Ann. Eugenics, 8, 52-56.

Wilk, M.B., Gnanadesikan, R., and Huyett, M.J. (1962). Estimation of parameters of the gamma distribution using order statistics. Biometrika, 49, 525-545.

Wright, F.T., Engelhardt, M. and Bain, L.J. (1978). Inferences for the two parameter exponential distribution under Type I censored sampling. J. Amer. Statist. Assoc., 73, 650-655.

BIOGRAPHICAL SKETCH

Lily Llorens Mantelle was born and reared in Santurce, Puerto Rico. She received the degree of Bachelor in Science with honors in mathematics from the University of Puerto Rico in 1980. She later attended the University of Florida in Gainesville, where she received her Master's degree in Statistics in May 1982. In December 1986 she expects to graduate with the degree of Doctor of Philosophy in Statistics from the same institution. She worked as a teaching assistant for various undergraduate courses while at the University of Florida and as a part-time statistical consultant for Key Pharmaceuticals in Miami, Florida from October 1983 until November 1984.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Malay Ghosh

Malay Ghosh, Chairman
Professor of Statistics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Kenneth M. Portier,
Associate Professor
of Statistics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Boghos D. Sivazlian,
Professor of Industrial and
Systems Engineering

This dissertation was submitted to the Graduate Faculty of the Department of Statistics in the College of Liberal Arts and Sciences and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

December 1986

Dean, Graduate School